Calculus IV Exam 1 Fall 1999 9/16/99

Each problem is worth 10 points. Show all work for full credit. Please circle all answers and keep your work as legible as possible. Nothing on this exam shall constitute grounds for litigation.

1. Find $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2+xy+y^2}$ or show that the limit does not exist.

Approach y axis, $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{x^2+y^2}{x^2+xy+y^2} = \lim_{y\to 0} \frac{0+y^2}{0+0+y^2} = 1$$

Approach $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x^2+y^2}{x^2+xy+y^2} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

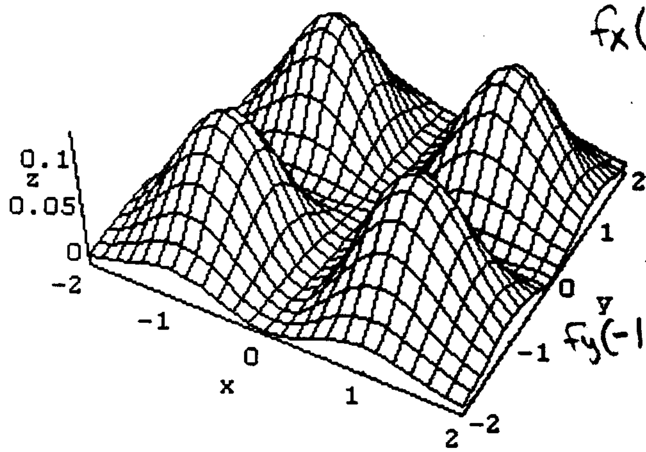
Approach $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{x^2+y^2}{x^2+xy+y^2} = \lim_{x\to 0} \frac{x^2+x^2}{x^2+x^2+x^2} = \lim_{x\to 0} \frac{2\cancel{x^2}}{3\cancel{x^2}} = 2/3$$

$\therefore$ limit does not exist b/c our approaches did not equal each other $1 \neq 2/3$

2. A flea named Bitsy is sitting at the point (-1,2) on the surface $g(x,y) = x^2y^2e^{-x^2-y^2}$. a) Which direction should Bitsy face in order to find the greatest uphill slope? b) How steep will the slope be in that direction?

W



$$f_x = 2xy^2e^{-x^2-y^2} - 2x^3y^2e^{-x^2-y^2}$$

$$f_x(-1,2) = 2(-1)(2)^2e^{-(-1)^2-(2)^2} - (2)(-1)(4)e^{-1-4}$$

$$= -8e^{-5} + 8e^{-5}$$

$$= 0$$

$$f_y = 2yx^2e^{-x^2-y^2} + x^2y^2e^{-x^2-y^2}(-2y)$$

$$f_y(-1,2) = 2(2)(-1)^2e^{-5} + (-1)^2(2)^2e^{-5}(-2)(2)$$

$$= 4e^{-5} - 16e^{-5}$$

$$= -12e^{-5}$$

a) $\boxed{\nabla f = \langle 0, -12e^{-5} \rangle}$

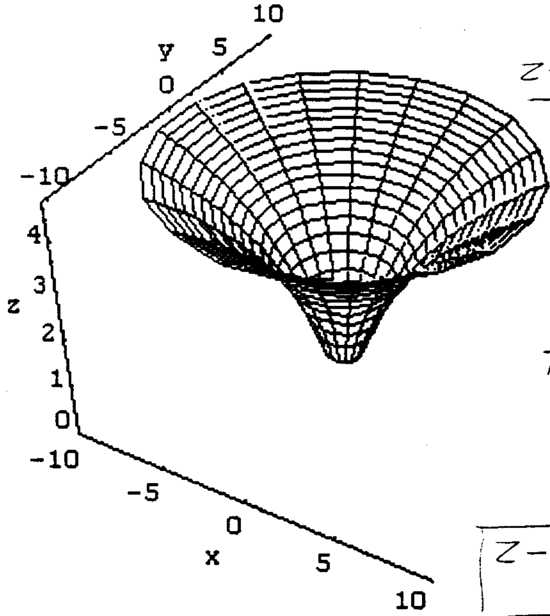
direction for greatest uphill slope

Very nice

b) steepness $= \sqrt{\left(\frac{-12}{e^5}\right)^2 + 0^2} = \boxed{\frac{12}{e^5}}$ steepness

10

3. Find the equation of the plane tangent to the surface $h(x,y) = \ln(x^2 + y^2 - 1)$ at the point $(-1,e)$.



general form of eqn. of Tangent plane:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$h_x = \frac{1}{(x^2+y^2-1)} \cdot 2x = \frac{2x}{(x^2+y^2-1)} = \frac{2(-1)}{(-1)^2+e^2-1} = -2/e^2$$

$$h_y = \frac{1}{(x^2+y^2-1)} \cdot (2y) = \frac{2y}{(x^2+y^2-1)} = \frac{2e}{(-1)^2+e^2-1} = \frac{2}{e}$$

To get z_0,

$$z_0 = \ln((-1)^2 + e^2 - 1)$$

$$z_0 = 2 \quad \therefore \text{ point is } (-1, e, 2)$$

$$\boxed{z - 2 = \frac{-2}{e^2} \cdot (x+1) + \frac{2}{e}(y-e)}$$

Excellent

10

4. It is regarded by many as a cause for severe national alarm that the price of gizmos has been increasing recently. One very recent study models the wholesale price of a ton of gizmos by the function $p(x,y,z)=5xy-3y^2z+15000$, where the mysterious factors x, y, and z all vary over time with current market research suggesting that $\frac{dx}{dt}=-1$, $\frac{dy}{dt}=3$, and $\frac{dz}{dt}=2$, while currently x=75, y=5, and z=9. Find $\frac{dp}{dt}$ based on current conditions.

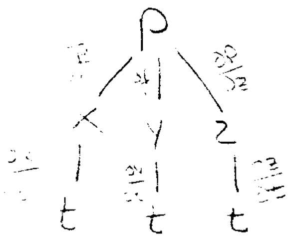
$p(x,y,z) = 5xy - 3y^2z + 1500$

$\frac{\partial p}{\partial x} = 5y \quad = 5(5) = 25$

$\frac{\partial p}{\partial y} = 5x - 6yz = 5(75) - 6(5)(9)$

$= 105$

$\frac{\partial p}{\partial z} = -3y^2 \quad = -75$



Great

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial p}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial p}{\partial z}\cdot\frac{\partial z}{\partial t}$$

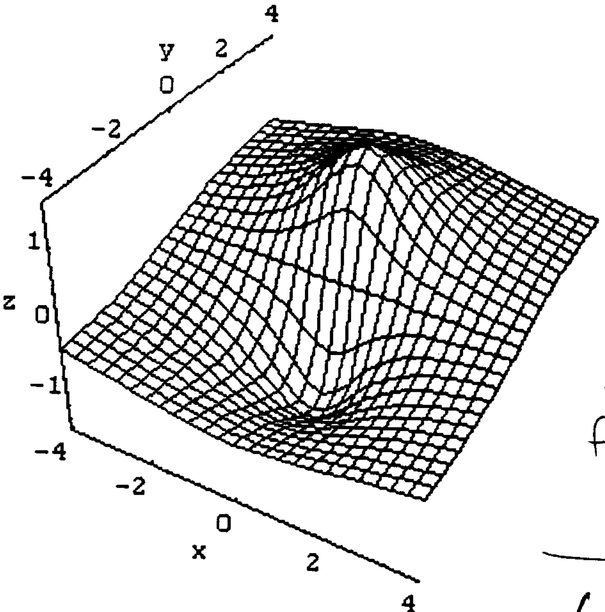
$$\frac{\partial p}{\partial t} = 25\cdot -1 + 105\cdot 3 + -75\cdot 2$$

$$= -25 + 315 + -150$$

$$\boxed{= 140}$$

Used (−), not (+).

5. Find all critical points of the function $f(x,y) = \frac{-3y}{x^2+y^2+1}$ and classify them as minima, maxima, saddle points or otherwise (you have to show which is which, the picture doesn't count as proof).



W

$$f_x(x,y) = \frac{\cancel{(x^2+y^2+1)(0)} - (-3y)(2x)}{(x^2+y^2+1)^2}$$

$$f_y(x,y) = \frac{(x^2+y^2+1)(-3) - (-3y)(2y)}{(x^2+y^2+1)^2}$$

$$f_x(x,y) = \frac{6xy}{(x^2+y^2+1)^2} \qquad f_y(x,y) = \frac{3y^2-3x^2-3}{(x^2+y^2+1)^2}$$

$$\frac{6xy}{(x^2+y^2+1)^2} = 0 \qquad \frac{3y^2-3x^2-3}{(x^2+y^2+1)^2} = 0$$

X or y must = 0

If x = 0, y = ±1
If y = 0: no x values [illegible]

Very nice

Crit. points = (0,1) & (0,−1)

$$f_{xx} = \frac{(x^2+y^2+1)^2(6y) - (6xy)[2(x^2+y^2+1)(2x)]}{(x^2+y^2+1)^4}$$

$f_{xx}(0,1) = \frac{24-0}{16} = \frac{3}{2}$

$f_{xx}(0,-1) = \frac{-24}{16} = -3/2$

$$f_{yy} = \frac{(x^2+y^2+1)^2(6y) - (3y^2-3x^2-3)[2(x^2+y^2+1)(2y)]}{(x^2+y^2+1)^4}$$

$f_{yy}(0,1) = \frac{24-0}{16} = 3/2$

$f_{yy}(0,-1) = \frac{-24-0}{16} = -3/2$

$$f_{xy}(x,y) = \frac{(x^2+y^2+1)^2(6x) - (6xy)[2(x^2+y^2+1)(2y)]}{(x^2+y^2+1)^4}$$

$f_{xy}(0,1) = 0$

$f_{xy}(0,-1) = 0$

$D = f_{xx}f_{yy} - f_{xy}^2$

for (0,1)

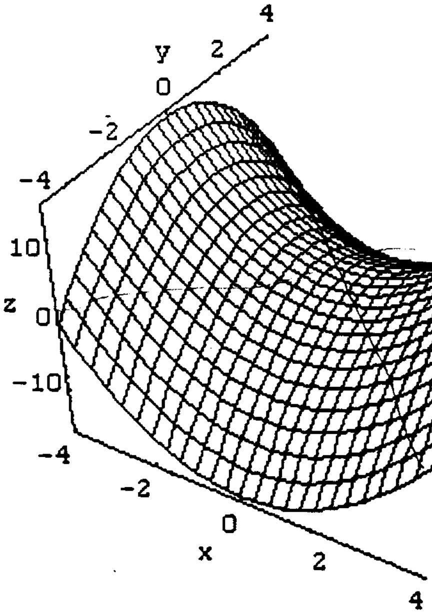
$D = (\frac{3}{2})(\frac{3}{2}) - 0 = \frac{9}{4}$ max or min, $f_{xx} = \frac{3}{2}$ so $\boxed{\text{Min.} = (0,1)}$

(0,−1)

$(-3/2)(-3/2) - 0 = \frac{9}{4}$ [illegible]

6. Suppose a dormouse is running along the surface $z = x^2 - y^2$ following the line $y = -x$. Show that the slope of the surface along the path the dormouse is following is zero.

Love it



W

→ Mouse running tangent to the surface.

Slope at any point on surface grad z

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = -2y$$

$$\nabla z = \langle 2x, -2y \rangle$$

unit vector in direction of mouse movement

$$u = \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

gradient at location of mouse is based on line $y = -x$

$$\nabla z_{(x_0,-x_0)} = \langle 2x_0, 2x_0 \rangle$$

magnitude of slope in direction of mouse movement

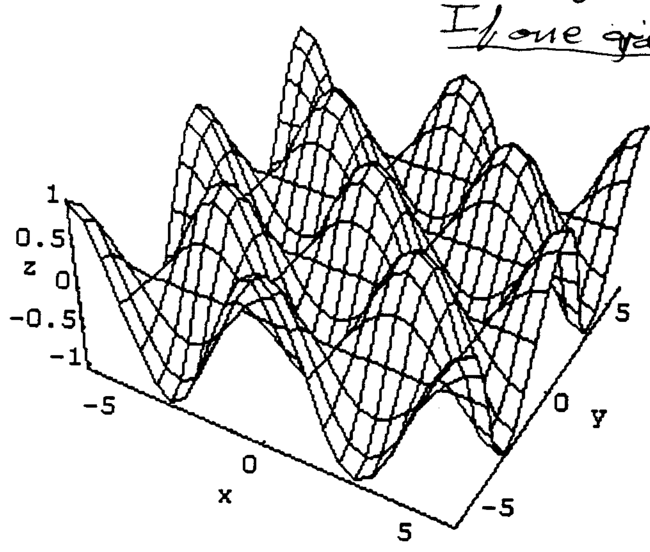
$$= \langle 2x_0, 2x_0 \rangle \left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{2x_0}{\sqrt{2}} - \frac{2x_0}{\sqrt{2}} = 0$$

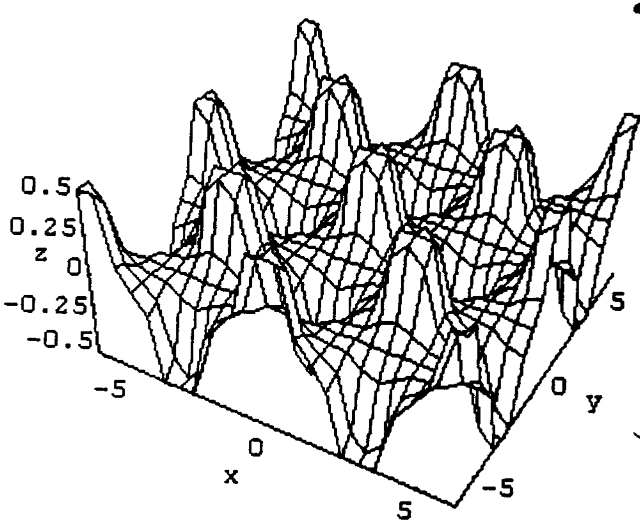
∴ mouse always experiences a zero slope on it's journey.

Great

7. The functions $f_1(x,y) = \cos x \cos y$ and $f_2(x,y) = \cos^3 x \cos^3 y$ look slightly similiar but not identical. Show that at any point (x,y), the gradients on both surfaces point in the same direction.



cos x cos y



$\cos^3 x \cos^3 y$

If one gradient is a scalar multiple of the other, they are parallel:

If $\nabla f_1(x_0,y_0) = \lambda \nabla f_2(x_0,y_0)$

then $\nabla f_1(x_0,y_0) \parallel \nabla f_2(x_0,y_0)$

$\nabla f_1 = \langle -\sin x \cos y, -\sin y \cos x \rangle$

$\nabla f_2 = \langle -3\cos^2 x \sin x \cos^3 y,$

$-3\cos^2 y \sin y \cos^3 x \rangle$

$= +3\cos^2 x \cos^2 y \langle -\sin x \cos y, -\sin y \cos x \rangle$

$\therefore \nabla f_2(x,y) = 3\cos^2 x \cos^2 y \, \nabla f_1(x,y)$

$\Rightarrow \boxed{\nabla f_1(x,y) = \lambda \nabla f_2(x,y)}$ where $\lambda = \frac{1}{3\cos^2 x \cos^2 y}$

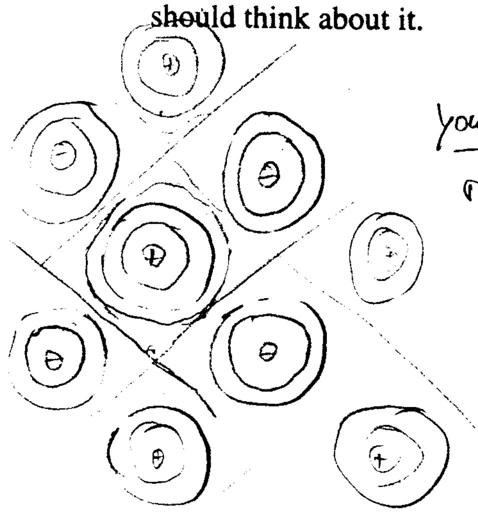
λ depends on $x + y_0$ but this changes only the ratio of the magnitudes.

10

Excellent

8. Biff and Buffy are Calculus students at O.S.U. who just got done taking their first exam and are arguing about one of the questions. Biff says "Uh, like, you know that one problem, the one about whether, uh, the level curves could cross? I said no 'cause then where they cross would be two different heights, right?" Buffy responds "Oh my *God*, like, that was *so* confusing. I like, said that they could, like, cross, you know? Because, you know, like, if there were, well I don't want to say it, but if there were like two bumps next to each other, then like the outline could be like a figure eight at, like, just the right height, you know? So like that would be, like, the level curve thingy would cross itself, wouldn't it? Oh my *God*."

Explain as clearly as possible to Buffy and Biff which of them (if either) is right and how they should think about it.



Well Buffy and Biff, I don't know exactly how your question was worded, but I think you are both right, in a way. You are thinking of two different things. Biff is thinking of the level curves for two different heights, or z-values. He is correct in saying they can't cross, for that would be an impossibility for a function of x and y; two z-values at the same (x,y). Buffy is thinking of the level curve for a certain height, and her bump example is on the right track. I've drawn a rough example of a surface with alternating positive and negative bumps. The crossing lines are where $z=0$. The grid is the level curve at $z=0$. It crosses itself many times, but with no contradictions. The slope along those lines is zero. Notice how none of the circular curves cross each other. Well, does that help?

Very nice

19. Show that the equation of the tangent plane to the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ at the point (x_0,y_0,z_0) can be written as $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-\frac{zz_0}{c^2}=1$.

10

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\frac{\partial z}{\partial x}: \frac{2x}{a^2}-\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \rightarrow -\frac{2z}{c^2}\frac{\partial z}{\partial x}=-\frac{2x}{a^2} \rightarrow \frac{\partial z}{\partial x}=\frac{\cancel{2}x}{a^2}\cdot\frac{c^2}{\cancel{2}z}$$

$$\frac{\partial z}{\partial x}=\frac{c^2x}{a^2z}$$

$$\frac{\partial z}{\partial y}: \frac{2y}{b^2}-\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \rightarrow -\frac{2z}{c^2}\frac{\partial z}{\partial y}=-\frac{2y}{b^2} \rightarrow \frac{\partial z}{\partial y}=\frac{\cancel{2}y}{b^2}\cdot\frac{c^2}{\cancel{2}z}$$

$$\frac{\partial z}{\partial y}=\frac{c^2y}{b^2z}$$

$$\rightarrow z-z_0=\frac{c^2x_0}{a^2z_0}(x-x_0)+\frac{c^2y_0}{b^2z_0}(y-y_0)$$

$$=\left(z-z_0=\frac{c^2x_0}{a^2z_0}(x-x_0)+\frac{c^2y_0}{b^2z_0}(y-y_0)\right)\cdot\frac{z_0}{c^2}$$

$$=\frac{z_0(z-z_0)}{c^2}=\frac{x_0(x-x_0)}{a^2}+\frac{y_0(y-y_0)}{b^2}$$

$$=\frac{zz_0}{c^2}-\frac{z_0^2}{c^2}=\frac{xx_0}{a^2}-\frac{x_0^2}{a^2}+\frac{yy_0}{b^2}-\frac{y_0^2}{b^2}$$

$$=\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-\frac{zz_0}{c^2}=\underbrace{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}}_{1}$$

Good

$$\boxed{\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-\frac{zz_0}{c^2}=1}$$

Nice

10. Suppose you are given that $D_u f(x_0,y_0) = 1$ and $D_v f(x_0,y_0) = 2$ for directional derivatives of a function in two different directions $\mathbf{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$ and $\mathbf{v} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$. Find an expression for the gradient of f at (x_0,y_0).

10

$D_u = \langle f_x, f_y \rangle \cdot \vec{u}$

$1 = \langle f_x, f_y \rangle \cdot \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$1 = \frac{1}{2} f_x + \frac{\sqrt{3}}{2} f_y$

$D_v = \langle f_x, f_y \rangle \cdot \vec{v}$

$2 = \langle f_x, f_y \rangle \cdot \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$2 = \frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y$

2 equations, 2 unknows: Solve!

$-\sqrt{3}\left[1 = \frac{1}{2} f_x + \frac{\sqrt{3}}{2} f_y\right] \Rightarrow -\sqrt{3} = -\frac{\sqrt{3}}{2} f_x - \frac{3}{2} f_y$

$2 = \frac{\sqrt{3}}{2} f_x + \frac{1}{2} f_y$

$2 - \sqrt{3} = -f_y$

$\sqrt{3} - 2 = f_y$

$1 = \frac{f_x}{2} + \frac{\sqrt{3}(\sqrt{3}-2)}{2}$

$1 = \frac{f_x}{2} + \frac{3 - 2\sqrt{3}}{2}$

$1 - \left(\frac{3 - 2\sqrt{3}}{2}\right) = \frac{f_x}{2}$

$\frac{2 - 3 + 2\sqrt{3}}{2} = \frac{f_x}{2}$

$2\sqrt{3} - 1 = f_x$

$\boxed{\nabla f = \langle 2\sqrt{3} - 1, \sqrt{3} - 2 \rangle}$

Extra Credit (5 points possible):

Find the maximum and minimum values, if any, of the function $f(x,y) = x^2 + y^2$ subject to the constraint $y = mx + b$ (in terms of m and b). [Recall that in class we did this with the constraint y=-x+4. If you're having trouble doing it in general, try one particular constraint line like y=2x+5.]

$f = \lambda \nabla g$

$g = c$

+5

$\nabla f = \langle 2x, 2y \rangle$ $\quad -b = mx - y$ $\quad \nabla g = \langle m, -1 \rangle$

$\nabla f = \lambda \nabla g$

$\langle 2x, 2y \rangle = \lambda \langle m, -1 \rangle$

$2x = \lambda m$ $\quad 2y = -\lambda$

$\frac{2x}{m} = \lambda$ $\quad -2y = \lambda$

$\frac{2x}{m} = -2y$

$2x = -2my$

$x = -my$

$-b = mx - y$

$-b = m(-my) - y$

$-b = -m^2 y - y$

$-b = y(-m^2 - 1)$

$\frac{b}{m^2+1} = y$

$x = \frac{-mb}{m^2+1}$

Critical Points

$$\left(\frac{-mb}{m^2+1}, \frac{b}{m^2+1}\right)$$

$$f(x,y) = x^2+y^2$$

$$f(x,y) = \frac{m^2b^2}{(m^2+1)^2} + \frac{b^2}{(m^2+1)^2}$$

$$= \frac{b^2(m^2+1)}{(m^2+1)^2}$$

$$= \boxed{\frac{b^2}{m^2+1} \leftarrow \text{The max + min values}}$$