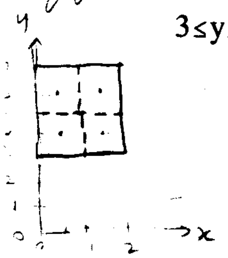
Calculus IV Exam 2 Fall 1999 10/21/99

Each problem is worth 10 points. Show all work for full credit. Please circle all answers and keep your work as legible as possible. This test has no adverse ecological impact on threatened species.

W

1. Use a double Riemann sum to approximate the value of $\iint_R \sqrt{x^3+\sqrt{y}}\,dA$ for $R=\{(x,y)\mid 0\le x\le 2,\ 3\le y\le 7\}$ using the partition lines $x=1$ and $y=5$ and letting (x_{ij}^*, y_{ij}^*) be the center of R_{ij}.



Coordinate of x_{ij}^*, y_{ij}^* = $\{(\tfrac{1}{2},4),(\tfrac{1}{2},6),(\tfrac{3}{2},4),(\tfrac{3}{2},6)\}$ $\Delta A = 1\times 2 = \underline{2}$

$$\text{Ans} = f(\tfrac{1}{2},4)2+f(\tfrac{1}{2},6)2+f(\tfrac{3}{2},4)2+f(\tfrac{3}{2},6)2$$

$=$

$f(\tfrac{1}{2},4) = \sqrt{(\tfrac{1}{2})^3+\sqrt{4}} = \sqrt{2.125} = 1.46$

$(\tfrac{1}{2},6) = \sqrt{(\tfrac{1}{2})^3+\sqrt{6}} = \sqrt{2.57} = 1.6$

$(\tfrac{3}{2},4) = \sqrt{(\tfrac{3}{2})^3+\sqrt{4}} = \sqrt{5.375} = 2.32$

$(\tfrac{3}{2},6) = \sqrt{(\tfrac{3}{2})^3+\sqrt{6}} = \sqrt{5.82} = 2.41$

$$\text{Ans} = (1.46)\cdot 2+(1.6)2+(2.32)2+(2.41)2$$

$$= \underline{15.58}$$

2. **Write** an iterated integral (in your choice of coordinate systems) with appropriate limits to integrate a function f over each of the following regions:

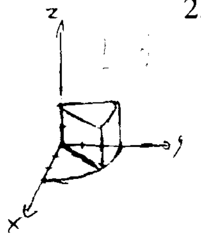
(a) The solid region between a sphere (centered at the origin) of radius 3 and a sphere of radius 5 below the plane z=0.

$x^2 + y^2 + z^2 = 3 \qquad x^2 + y^2 + z^2 = 5$

$$\boxed{\int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \int_3^5 f \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.}$$

(b) A wedge of cheddar cheese sitting on the xy plane in the first octant with radius 3 and height 2, with one face on the plane x=0 and the angle between the faces π/6.

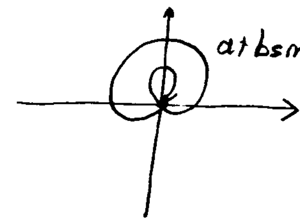
$= \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$



$$\boxed{V = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^3 \int_0^2 1 \cdot r \, dz \, dr \, d\theta}$$

3. Suppose you're walking down the street one day and you find an arbitrarily thin, flat sheet of some rigid slug-colored substance shaped like the **inner loop** of the limaçon r=1+2sinθ. **Set up** a double integral for the area of this object.

The thing is already in polar coordinates, so I use them to integrate.



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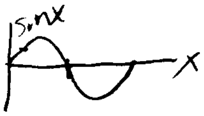
$A = \int\int_R r\,dr\,d\theta$, since $J = r$



I recognize that the inner loop on this graph touches the origin as shown, implying $r=0$ at some point.

$0 = 1 + 2\sin\theta \rightarrow -\frac{1}{2} = \sin\theta \rightarrow \theta = -\frac{\pi}{6}, \frac{7\pi}{6} = \theta$. I choose to avoid using $\theta < 0$.

On $[0, 2\pi]$, $\frac{7\pi}{6} = \theta$ and $\frac{11\pi}{6} = \theta$ touch the origin. R varies from 0 to $1 + 2\sin\theta$, so

$$A = \int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} \int_0^{1+2\sin\theta} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{\frac{7\pi}{6}}^{\frac{11\pi}{6}} (1+2\sin\theta)^2\,d\theta.$$

Well done.

4. Find the surface area of the portion of the cone $z^2 = x^2 + y^2$ which lies within the cylinder $x^2 + y^2 = 9$.

gradient

W

limits

$y = \pm\sqrt{9 - x^2}$

x (when $y=0$) $= \pm\sqrt{9}$
$= \pm 3$

going in complete 2π

$\theta =$

$z = \sqrt{x^2 + y^2}$

$\dfrac{\partial z}{\partial x} = \dfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x = \dfrac{x}{\sqrt{x^2+y^2}}$

$\dfrac{\partial z}{\partial y} = \dfrac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y = \dfrac{y}{\sqrt{x^2+y^2}}$

$\left(\dfrac{\partial z}{\partial x}\right)^2 = \dfrac{x^2}{x^2+y^2}$

$\left(\dfrac{\partial z}{\partial y}\right)^2 = \dfrac{y^2}{x^2+y^2}$

Excellent

$SA = \iint_D \sqrt{\partial x^2 + \partial y^2 + 1}\; dA$

$SA = \iint_D \sqrt{\dfrac{x^2}{x^2+y^2} + \dfrac{y^2}{x^2+y^2} + 1}\; dA$

$SA = \displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{\dfrac{x^2+y^2+x^2+y^2}{x^2+y^2}}\; dy\,dx$

$SA = \displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{\dfrac{2x^2+2y^2}{x^2+y^2}}\; dy\,dx$

* looks like a circle so, I'm going to switch to polar coordinates

$y = \sqrt{9-x^2}$ $\quad y^2 + x^2 = 9$ $\quad r^2 = 9$ $\quad r = 3$

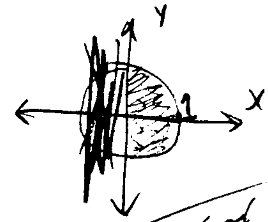
$SA = \displaystyle\int_0^{2\pi}\int_0^{3} \sqrt{\dfrac{2r^2}{r^2}}\; r\,dr\,d\theta$

$SA = \displaystyle\int_0^{2\pi}\int_0^{3} \sqrt{2}\; r\,dr\,d\theta$

$SA = \displaystyle\int_0^{2\pi} \sqrt{2}\, \tfrac{1}{2} r^2 \Big|_0^3\, d\theta = \int_0^{2\pi} \sqrt{2} \cdot \tfrac{9}{2}\, d\theta$

$= \cancel{2}\pi \cdot \sqrt{2} \cdot \dfrac{9}{\cancel{2}} = \pi \cdot \sqrt{2} \cdot 9$

5. Compute the integral $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{e^{-(x^2+y^2)}} 1\, dz\, dy\, dx$



Good

X

Convert to cylindrical coordinates:

W $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^1 \int_0^{e^{-r^2}} r \; dz\, dr\, d\theta$

$\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^1 \left[rz \right]_0^{e^{-r^2}} dr\, d\theta$

$\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^1 r(e^{-r^2}) \; dr\, d\theta$

$\displaystyle\int_{-\pi/2}^{\pi/2} \left[-\frac{1}{2}\left(e^{-r^2}\right) \right]_0^1 d\theta$

$(-\frac{1}{2}e^{-1}) - (-\frac{1}{2}\cdot 1)$

$\displaystyle\int_{-\pi/2}^{\pi/2} \left(-\frac{1}{2}e^{-1} + \frac{1}{2}\right) d\theta$

$= \left[-\frac{1}{2}e^{-1}\theta + \frac{1}{2}\theta \right]_{-\pi/2}^{\pi/2}$

$= \left[-\frac{1}{2}e^{-1}(\pi/2) + \frac{1}{2}\left(\frac{\pi}{2}\right) \right] - \left[-\frac{1}{2}e^{-1}(-\pi/2) + \frac{1}{2}(-\pi/2) \right]$

$= -\frac{1}{4}e^{-1}(\pi) + \frac{1}{4}\pi - \frac{1}{4}e^{-1}(\pi) + \frac{1}{4}\pi$

$= -\frac{1}{2}e^{-1}(\pi) + \frac{\pi}{2}$

Great

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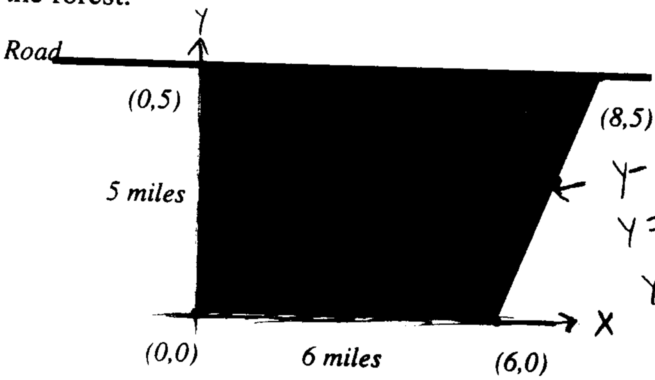
6. A forest next to a road has the shape shown below. The population density of marmots in the forest is given by D(x,y)=10-2y marmots per square mile. Find the total marmot population in the forest.



$y - 0 = \left(\frac{5-0}{8-6}\right)(x-6)$

$y = \frac{5}{2}(x-6)$

$y = \frac{5}{2}x - 15$

$x = \frac{2y}{5} + 6$

Vancouver Marmot

$$\int_0^5 \int_0^{\frac{2y}{5}+6} (10-2y)\, dx\, dy$$

$$= \int_0^5 \left[-2xy + 10x\right]_0^{\frac{2y}{5}+6} dy$$

$$= \int_0^5 \left[\frac{-4y^2}{5} - 8y + 60\right] dy = \left[\frac{-4y^3}{15} - 4y^2 + 60y\right]_0^5$$

$$= \frac{500}{3}$$

Nice

7. Biff says "Uh, so my Calc professor said I was wrong on this thing on the test. What happened was I got a negative number for the volume of this thing, so I just took off the negative sign and circled that. I mean, if you get a negative for volume it must have just been the opposite of the right answer, right? But she said, uh, like this confusing stuff that I didn't understand, and said that wasn't right. But she said I didn't screw up working it out, so I think she just hates me because I found a shortcut."

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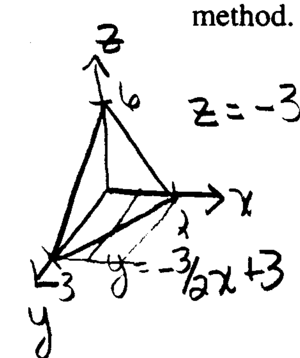
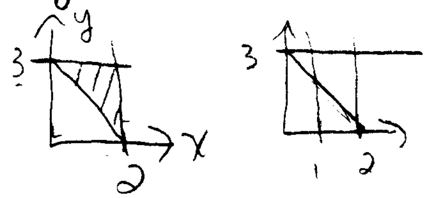
Either (if you think Biff's method is right) explain to Biff's professor why what he did is valid, or (if you think Biff's professor is right) explain to Biff what goes wrong with his "shortcut" method.



If Biff were integrating the region bounded by the plane $z=-3x-2y+6$, $y=-\frac{3}{2}x+3$, $y=3$, and $x=2$ then his probem would look like this:

$$\int_0^2 \int_{-\frac{3}{2}x+3}^{3} \int_0^{-3x-2y+6} 1 \, dz\, dy\, dx$$



$$\int_0^2 \int_{-\frac{3}{2}x+3}^{3} (-3x-2y+6)\, dy\, dx$$

$$\int_0^2 \left(-3xy-y^2+6y\right)\Big]_{y=-\frac{3}{2}x+3}^{3} dx$$

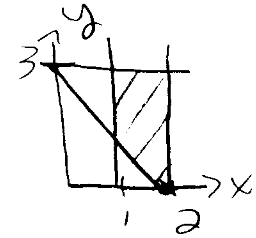
$$= \int_0^2 \left[(-9x-9+18) - \tfrac{9}{2}x^2 + 9x + \left(\tfrac{9}{4}x^2 - 9x + 9\right) + 9x - 18)\right] dx$$

$$= \int_0^2 \left(-\frac{9}{2}x^2\right) dx = \left[-\frac{3}{2}x^3\right]_{x=0}^{2} = \underline{-12}$$

In this case, Biff can just take away the negative sign, and he will have the volume. BUT this only works because of the symmetry of the problem. The region he integrated is identical to the region bounded by $z=-3x-2y+6$, $y=-\frac{3}{2}x+3$, $y=0$, and $x=0$. Integrating that region would produce $+12$. If the region he integrated on were not so cut & dry, then he could not just take away the negative sign. For example, if he were integrating the region bounded by $z=-3x-2y+6$,

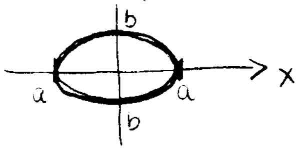
$x=1$, $x=2$, $y=3$, and $y=0$ he might set up an integral like this.

$$\int_1^2 \int_0^3 \int_0^{-3x-2y+6} dz\,dy\,dx$$



This would produce a negative answer because the plane is below the x-y plane in most of this region. Just removing the (-) sign would NOT give you the volume though. It would give you the difference of the volume for the shape above the x-y plane and the shape below it.

Excellent



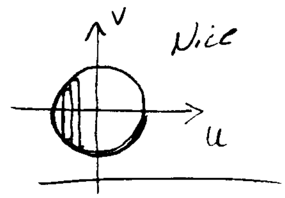
8. Use the transformation x=au, y=bv to compute $\iint_R 1\,dA$ for the region R bounded by the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The Jacobian $= \left\| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{matrix} \right\| = \left\| \begin{matrix} a & 0 \\ 0 & b \end{matrix} \right\| = ab$

$$\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} 1\,(ab)\,dv\,du$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow \frac{\cancel{a^2}u^2}{\cancel{a^2}}+\frac{b^2v^2}{\cancel{b^2}}=1$$

$$u^2+v^2=1$$



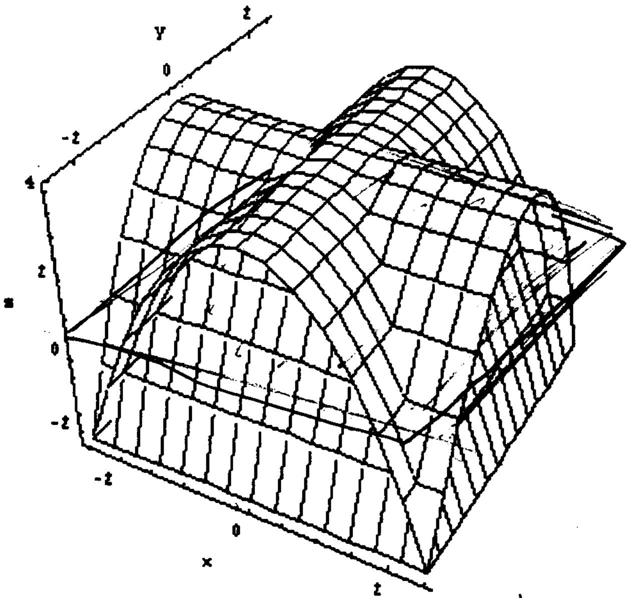
Nice

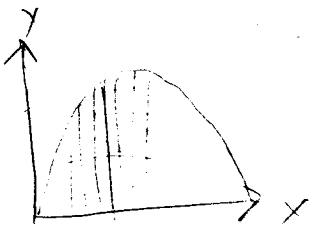
$$= \int_0^{2\pi}\int_0^1 ab\,r\,dr\,d\theta = ab\int_0^{2\pi}\tfrac{1}{2}\left[r^2\right]_0^1 d\theta$$

$$= ab\int_0^{2\pi}\tfrac{1}{2}(1)\,d\theta = \frac{ab}{2}\left[\theta\right]_0^{2\pi} = \frac{ab(\cancel{2}\pi)}{\cancel{2}} = \boxed{\pi ab}$$

Excellent

10. Find the volume of the region above the plane z=0 and below both the surface $z=4-x^2$ and the surface $z=4-y^2$





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$V = \iiint dy\,dx\,dz$

$-x^2 = z-4$

$x^2 = 4-z$

$x = \pm\sqrt{4-z}$

$y = \pm\sqrt{4-z}$

Limits

$x \to -2 \le x \le 2$

$y \to -2 \le y \le 2$

$\boxed{z \to 0 \le z \le 4}$

Excellent

$$\int_0^4 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} \int_{-\sqrt{4-z}}^{\sqrt{4-z}} dy\,dx\,dz = \int_0^4 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} y \Big|_{-\sqrt{4-z}}^{\sqrt{4-z}} dx\,dz = \sqrt{4-z} + \sqrt{4-z} = 2\sqrt{4-z}$$

$$\int_0^4 \int_{-\sqrt{4-z}}^{\sqrt{4-z}} 2\sqrt{4-z}\,dx\,dz = \int_0^4 2\sqrt{4-z}\,(x) \Big|_{-\sqrt{4-z}}^{\sqrt{4-z}} = \int_0^4 2(4-z) + 2(4-z)$$

$$= \int_0^4 16 - 4z\,dz = 16z - 2z^2 \Big|_0^4 = 64 - 32 - 0 = \boxed{32}$$

Extra Credit (5 points possible):

Generalize problem 10 to the situation where the two parabolic cylinders are $z=a-x^2$ and $z=b-y^2$ [if you can't do the whole thing, at least find the dimensions of the projection in the xy plane, i.e. the top view].