9/28/99 CALCULUS IV - PROBLEM SET 1

10/10

1. a) The direction of the normal line of F is given by ∇F and for G, it is given by ∇G. Assuming $\nabla F \neq 0 \neq \nabla G$ the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at P.

or $\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \left\langle \frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}, \frac{\partial G}{\partial z} \right\rangle = 0$ at P

or $F_x G_x + F_y G_y + F_z G_z = 0$ at P

Great

b) $F = x^2+y^2-a^2z^2$ and $G = x^2+y^2+z^2-r^2$

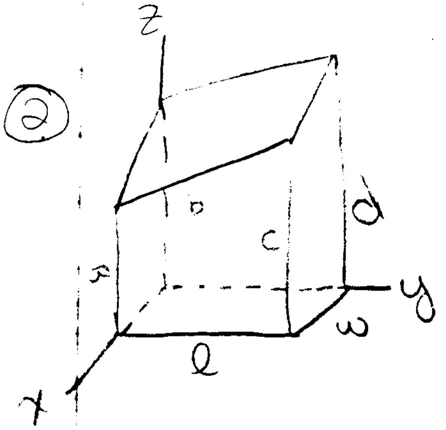
$$\nabla F \cdot \nabla G = \langle 2x, 2y, -2a^2z \rangle \cdot \langle 2x, 2y, 2z \rangle$$

$$= 4x^2+4y^2-4a^2z^2 = 4F = 0$$ since $\langle x, y, z \rangle$ lies on the graph of $F=0$

$F=0$ is a equation of a right circular cone with vertex at the origin and $G=0$ is a sphere centered at the origin. At any point of intersection the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F=0$ and $G=0$ are orthogonal everywhere.

Excellent

(2)



1) Find a pt on the plane

$P(x,y,z) = (0,0,b)$

2) Find another pt on the plane

$P_0(x,y,z) = (w,l,c)$

3) Find a 3rd 4th pt on the plane

$P_{00}(x,y,z) = (w,0,a)$

$P_{000}(x,y,z) = (0,l,d)$

4) Find a vector on the plane:

$\langle w, l, c-b \rangle$

5) Find another vector on the plane:

$\langle w, -l, a-d \rangle$

6) The cross product of these will give a normal vector to the plane

$\langle w, l, c-b \rangle \times \langle w, -l, a-d \rangle$

$\vec{n} = \langle l(a-b+c-d), w(-a-b+c+d), -2wl \rangle$

$\vec{u} = \langle x, y, z-b \rangle$

7) So, equation of the plane is $\vec{n} \cdot \vec{u} = 0$.

$l(a-b+c-d)x + w(-a-b+c+d)y - 2wlz + 2wlb = 0$

$l(a-b+c-d)x + w(-a-b+c+d) + 2wlb = 2wlz$

$$\boxed{\frac{a-b+c-d}{2w}x + \frac{-a-b+c+d}{2l}y + b = z}$$

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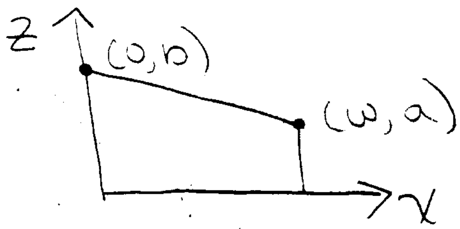
b) Find the x-intercept of the plane



Equation of Line:

$$z - z_0 = \frac{a-b}{w}(x - x_0)$$

$$z - b = \frac{a-b}{w}(x)$$

we want x-int. (when $z=0$)

$$-b = \frac{a-b}{w}x$$

$$\frac{-wb}{a-b} = x$$ ← This is the x-int of the plane

Using the equation of the plane where $y=0$ and $z=0$ we can find another representation of the x-int.

$$\frac{a-b+c-d}{2w}x + 0 + b = 0$$

$$\frac{a-b+c-d}{2w}x = -b$$

$$x = \frac{-2bw}{a-b+c-d}$$

- OVER -

Now, we set these equal.

$$\frac{\cancel{-wb}}{a-b} = \frac{\cancel{-2wb}}{a-b+c-d}$$

$$2a-2b=a-b+c-d$$

$$a+d=b+c$$

This is beautiful!

+1

$\boxed{\therefore \text{ a+d must always equal b+c}}$

c) $\int_0^{\ell}\int_0^{w} \left[\frac{a-b+c-d}{2w}x + \frac{-a-b+c+d}{2\ell}y + b\right] dx\,dy$

$= \int_0^{\ell} \left[\frac{a-b+c-d}{2w}\left(\frac{x^2}{2}\right) + \frac{-a-b+c+d}{2\ell}yx + bx\right]_{x=0}^{w} dy$

$= \int_0^{\ell}\left[\left(\frac{(a-b+c-d)w}{4}\right) + \left(\frac{-a-b+c+d}{2\ell}\right)wy + bw\right] dy$

$= \left[\frac{a-b+c-d}{4}wy + \frac{-a-b+c+d}{2\ell}\,\frac{wy^2}{2} + bwy\right]_{y=0}^{\ell}$

$= \frac{a-b+c-d}{4}w\ell + \frac{-a-b+c+d}{2\ell}\,\frac{w\ell^2}{2} + bw\ell$

$= \frac{a-b+c-d}{4}w\ell + \frac{-a-b+c+d}{4}w\ell + bw\ell$

$= w\ell\left(\frac{\cancel{a}-b+c\cancel{-d}\cancel{-a}-b+c+\cancel{d}+4b}{4}\right)$

$= w\ell\left(\frac{2b+2c}{4}\right) = w\ell\,\frac{(b+c)}{2}$

Excellent

The iterated integral is equal to:

$\frac{w\ell(b+c)}{2}$ or (using result from (b)) $\frac{w\ell(a+d)}{2}$