Calculus III Exam 2 Spring 2000 3/10/2000

Each problem is worth 10 points. Be sure to show all work and justifications for full credit. Please circle all answers and keep your work as legible as possible. No animals were used in the making of this exam.

1. *Set up* an integral for the length of one arch (i.e. for $0 \le t \le 2\pi$) of the cycloid with parametric equations $x(t)=t-\sin t$ and $y(t)=1-\cos t$.

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \frac{dx}{dt} = 1-\cos t \qquad \frac{dy}{dt} = \sin t$$

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$$= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt$$

2. Find the vertex, focus, and directrix of the parabola $x^2 = -12y$ and sketch its graph, indicating the positions of the focus and directrix on the graph.

$x^2 = -12y$ $\qquad x^2 = 4py$

$-12 = 4p$

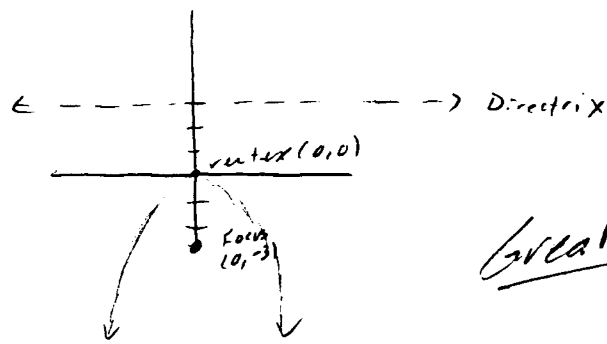
10

$-3 = p$

Focus = $(0,-3)$

Vertex = $(0,0)$

directrix = $y = 3$



Great

3. *Set up* an integral for the area inside the loop of the graph of the parametric equations $x(t)=t^2$, $y(t)=t^5-4t$.

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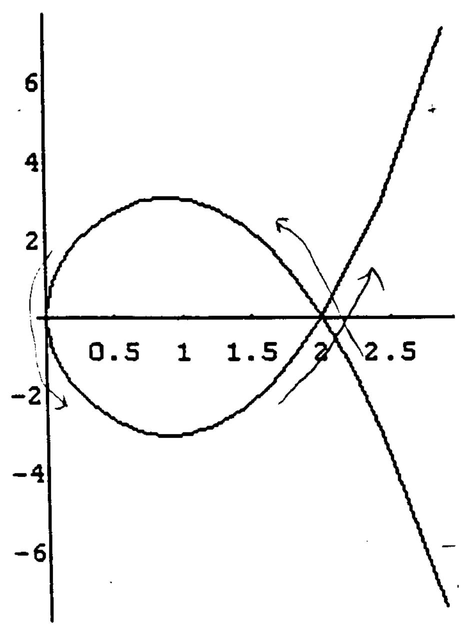


Graph for $-1.7 \le t \le 1.7$

X int when $y(t)=0$

$0 = t^5 - 4t \qquad t^4 - 4 = 0 \qquad (t^2-2)(t^2+2) = 0$

$t = 0, \sqrt{2}, -\sqrt{2}$

$y(t) = t^5 - 4t$

$x'(t) = 2t$

$\int_a^b y(t)x'(t)\, dt$

top half and bottom half are equal

$\boxed{-2\int_0^{\sqrt{2}} (t^5-4t)(2t)\, dt}$

Great

4. Show that the third degree MacLaurin polynomial for $f(x)=\tan^{-1}(x)$ is $x-\frac{x^3}{3}$ [Recall that the derivative of $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$].

$f(x) = \tan^{-1}(x)$ $\qquad f(0) = 0$

$f'(x) = \frac{1}{1+x^2}$ $\qquad f'(0) = 1$

$f''(x) = \frac{-2x}{(1+x^2)^2}$ $\qquad f''(0) = 0$

$f'''(x) = (-2x)(-2)(1+x^2)^{-1}(2x) - 2(1+x^2)^{-2}$ $\qquad f'''(0) = -2$

$0 + x + \frac{0}{2!}x^2 + \frac{-2}{3!}x^3 = x + \frac{-2x^3}{1 \cdot 2 \cdot 3} = \boxed{x - \frac{x^3}{3}}$

10

Nice

5. Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$.

11

Ratio Test $\left|\frac{a_{n+1}}{a_n}\right|$ $\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{2(n+1)+1}}{\frac{(-1)^n x^{2n+1}}{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}} x^{2n+3}}{2n+3} \cdot \frac{2n+1}{\cancel{(-1)^n} x^{2n+1}} \right|$

$\lim_{n\to\infty} \left| \frac{x^3(2n+1)}{x(2n+3)} \right| = x^2 \lim_{n\to\infty} \left| \frac{\frac{2n}{n}+\frac{1}{n}}{\frac{2n}{n}+\frac{3}{n}} \right| = \frac{2}{2} = 1$ $\qquad |x^2| < 1$

$R = 1$

Great

6. Given that $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ is the MacLaurin series for $f(x) = \tan^{-1}(x)$, use the fifth degree MacLaurin polynomial to approximate $f\left(\frac{\sqrt{3}}{3}\right)$ [realize that $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$, so six times your result should be a reasonably good approximation for pi].

10

$f\left(\frac{\sqrt{3}}{3}\right) \approx$

$\frac{\sqrt{3}/3}{1} - \frac{(\sqrt{3}/3)^3}{3} + \frac{(\sqrt{3}/3)^5}{5} = \boxed{.52603}$

$6 \cdot .52603 = 3.15619$

$\pi \approx 3.14159$

Great

7. Barbie is a Calculus student from California, and she's having some trouble. Barbie says "Power series are *hard*. We had to do this problem set thingy, and it said to , like, approximate pi with the series for inverse tangent, so my friend Skipper said we should do the inverse tangent of the square root of three, 'cause that would be pi over three and we could multiply it by three to get pi. But we keep doing, you know, more and more terms, but our answers didn't get very close to pi. So it's like, we had to do even more terms, I guess, to make it right. That's too much work, and Skipper and I wouldn't have enough time to go to the gym and tan."

10 Is Barbie right that more terms will improve their approximation? Explain to her why or why not.

Well Barbie, don't fret, you'll be able to tan soon. You see, MacLaurin series are power series, so they don't necessarily converge for all terms x. They have what is called a radius of convergence; a certain range over which the series approximation of the function works. If you plug in numbers outside of that radius of convergence, then the series will just get farther and farther from the desired value until you get to infinity. So, my dear, just give up.

Excellent explanation - terrible advice!

8. Sketch a graph of the conic section with equation $3y^2 + 6y + 7 = 16x - 4x^2$. State whether it is a hyperbola or ellipse and include on your sketch (if it's a hyperbola) the asymptotes and coordinates of points on the axis of symmetry or (if it's an ellipse) the coordinates of the four points on the major and minor axes.

$$3y^2 + 6y + 7 = 16x - 4x^2$$

$$3y^2 + 6y \quad + 4x^2 - 16x = -7$$

$$3(y^2 + 2y + 1) + 4(x^2 - 4x + 4) = -7 + 3 + 16$$

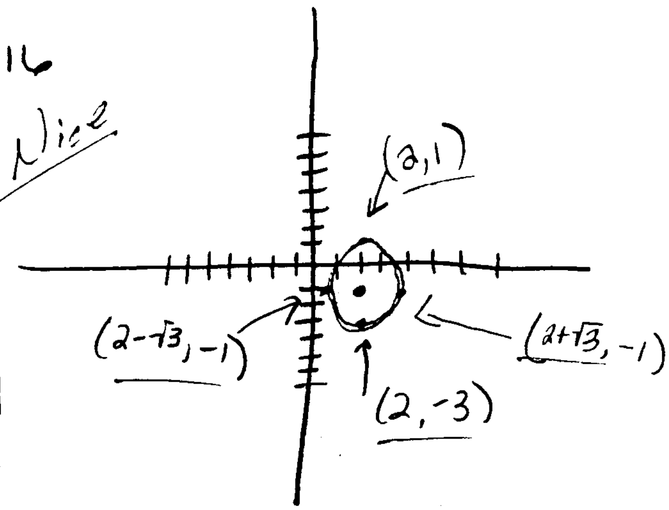
$$3(y+1)^2 + 4(x-2)^2 = 12$$

10 $\quad \dfrac{(y+1)^2}{4} + \dfrac{(x-2)^2}{3} = 1$

$(2,-1)$

Very Nice

ellipse!



9. For what value of c will the area inside $r=c+\cos\theta$ be the same as the area inside a circle of radius 1? [Hint: If you can't figure out how to go about it, you might start by finding the area inside $r=1+\cos\theta$ and see what you can figure from there. You might also benefit from using the integration formula $\int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$.]

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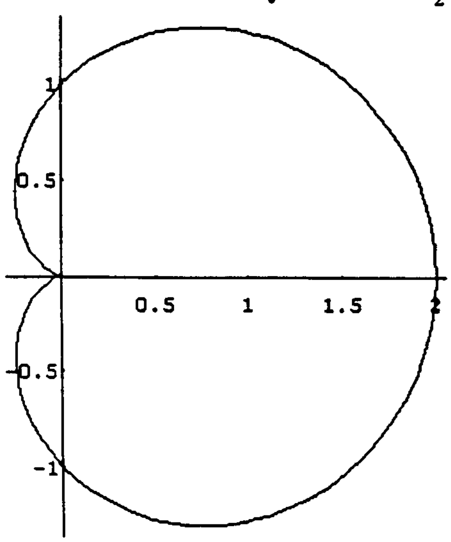


Graph of $1+\cos\theta$ for $0 \le \theta \le 2\pi$

Area for circle radius $2 = 4\pi$

$$4\pi = \frac{1}{2}\int_0^{2\pi} (c + \cos\theta)^2\, d\theta$$

$$4\pi = \int_0^{\pi} (c + \cos\theta)^2\, d\theta$$

$$4\pi = \int_0^{\pi} c^2 + 2c\cos\theta + \cos^2\theta\, d\theta$$

$$4\pi = c^2\theta\Big]_0^{\pi} + 2c\sin\theta\Big]_0^{\pi} + \frac{1}{2}\theta\Big]_0^{\pi} + \frac{1}{4}\sin 2\theta$$

$$0$$

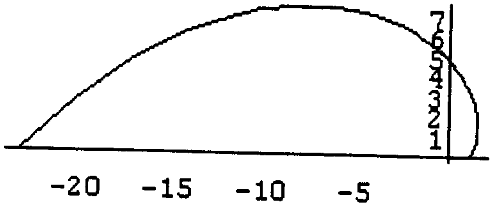
$$4\pi = c^2\pi + \quad + \frac{\pi}{2}$$

$$\frac{7\pi}{2} = c^2\pi \qquad c^2 = 7/2 \qquad \boxed{c = \pm\sqrt{7/2}}$$

Very nice

10. Consider the polar function $r(\theta)=e^{\theta}$. When does it have a horizontal tangent line? [Hint: For starters find the slope.]

W



Graph for $0\leq\theta\leq\pi$

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \quad \begin{array}{l} x = e^{\theta}\cos\theta \\ y = e^{\theta}\sin\theta \end{array} \Rightarrow \frac{(e^{\theta}\cdot\sin\theta)'}{(e^{\theta}\cdot\cos\theta)'}$$

$$\Rightarrow \frac{(e^{\theta})'\sin\theta + (e^{\theta})\cos\theta}{(e^{\theta})'\cos\theta - e^{\theta}\sin\theta} = \underline{\frac{e^{\theta}\sin\theta + e^{\theta}\cos\theta}{e^{\theta}\cos\theta - e^{\theta}\sin\theta}}$$

since $(e^{x})' = e^{x}$ . . .

Horizontal when $\underline{\frac{dy}{dt} = 0}$ $\Rightarrow$ $\underline{e^{\theta}\sin\theta + e^{\theta}\cos\theta = 0}$

$\sin\theta + \cos\theta = 0$

$\underline{\theta = \frac{-\pi}{4}}$ $\Rightarrow$ $\frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = 0$

This graph is horizontal when $\theta = \frac{-\pi}{4}$ Excellent - for one!

Extra Credit (5 points possible):

We dealt with several of the possible arrangements of distances to foci and directrices, but there are others that can give surprising results. Consider the collection of points whose distances from (2,0) are twice their distances from (-1,0). What does this collection look like? At the least, find a few points that satisfy this requirement.