PS1

#1 $\left\{ \frac{1}{n} - \frac{1}{n+1} \right\}$

$$a_1 = \frac{1}{1} - \frac{1}{2} = \frac{1}{2}$$

$$a_2 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

$$a_3 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$a_4 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

For the limit:

$$\lim_{n \to \infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{n \to \infty} \left(\frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} \right) = \lim_{n \to \infty} \frac{1}{n(n+1)} = 0$$

...and I notice that this resembles part of #5!

PS 1

#2

$$a_n = \frac{n^2}{n(2n-1)}$$

$$a_1 = \frac{1^2}{1(2-1)} = \frac{1}{1}$$

$$a_2 = \frac{2^2}{2(4-1)} = \frac{2}{3}$$

$$a_3 = \frac{3^2}{3(6-1)} = \frac{3}{5}$$

$$a_4 = \frac{4^2}{4(8-1)} = \frac{4}{7}$$

So for the limit:

$$\lim_{n\to\infty} \frac{n^2}{n(2n-1)} = \lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{n/n}{2-1/n} = \frac{1}{2}$$

PS 1

#3 $$\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$$

$$s_1 = \frac{2+1}{1^2(2)^2} = \frac{3}{4}$$

$$s_2 = \frac{3}{4} + \frac{4+1}{2^2(2+1)^2} = \frac{3}{4} + \frac{5}{36} = \frac{32}{36} = \frac{8}{9}$$

$$s_3 = \frac{3}{4} + \frac{5}{36} + \frac{6+1}{3^2(3+1)^2} = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} = \frac{15}{16}$$

$$s_4 = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{8+1}{4^2(4+1)^2} = \frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{9}{400} = \frac{24}{25}$$

To find the limit:

Using the partial fractions method from Calc II,

$$\frac{2n+1}{n^2(n+1)^2} = \frac{1}{n^2} - \frac{1}{(n+1)^2}$$

so when I add terms:

$$\left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \left(\frac{1}{4^2} - \frac{1}{5^2}\right) + \dots$$

I notice everything but the first $\frac{1}{1^2}$ and last $\frac{1}{(n+1)^2}$ cancels out, so the n^{th} partial sum $s_n = 1 - \frac{1}{(n+1)^2}$ and as n grows large this approaches 1.

PS 4

#4

I know $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges

And I know:
$\sin^2 n \leq 1$

so: $\frac{\sin^2 n}{n!} \leq \frac{1}{n!}$ and all the terms are positive so by comparison

$\sum \frac{\sin^2 n}{n!} \leq \sum \frac{1}{n!}$

and thus $\sum_{n=1}^{\infty} \frac{3\sin^2 n}{n!}$ converges

PS 1

#5 $\sum_{n=1}^{\infty}\left[\frac{4}{2^n} - \frac{2}{n(n+1)}\right]$

$s_1 = 2 - 1 = 1$

$s_2 = (2 - 1) + (1 - \frac{1}{3}) = 1\,^2/_3$

$s_3 = (2-1) + (1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{6}) = 2$

$s_4 = (2-1) + (1 - \frac{1}{3}) + (\frac{1}{2} - \frac{1}{6}) + (\frac{1}{4} - \frac{1}{10}) = 2\,^3/_{20}$

So for the limit:

I recognize the pattern of 2, 1, 1/2, 1/4 in the stuff above, so I can manage $\sum_{n=1}^{\infty}\frac{4}{2^n}$ as a geometric series with $a = 2$ and $r = 1/2$. This totals to $\frac{2}{1-1/2} = 4$.

As for $\frac{2}{n(n+1)}$, it resembles stuff that turned up in #1, so I take this clue to rewrite it as $\frac{2}{n} - \frac{2}{n+1}$. Then adding up terms I'll have

$$\left(\frac{2}{1} - \frac{2}{2}\right) + \left(\frac{2}{2} - \frac{2}{3}\right) + \left(\frac{2}{3} - \frac{2}{4}\right) + \left(\frac{2}{4} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{2}{6}\right) + \ldots$$

where everything but the first term $\frac{2}{1}$ and the last term cancel. So the n^{th} partial sum $s_n = 2 - \frac{2}{n+1}$ and as $n \to \infty$, $s_n \to 2$.

PS 1

#6 $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{e^{\sqrt{n}}}$ is cont., pos., and decreasing, so we can use the integral test.

$$\int \frac{\sqrt{x}}{e^{\sqrt{x}}}\,dx = \frac{-2x-4\sqrt{x}-4}{e^{\sqrt{x}}}+C$$ by an ugly process of substitution and integration by parts, so we look at

$$\lim_{b\to\infty} \int_1^b \frac{\sqrt{x}}{e^{\sqrt{x}}}\,dx = \lim_{b\to\infty}\left[\frac{-2x-4\sqrt{x}-4}{e^{\sqrt{x}}}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{-2b-4\sqrt{b}-4}{e^{\sqrt{b}}} - \frac{-2-4-4}{e^1}\right]$$

$$= 0 + \frac{10}{e}$$ by L'Hopital's Rule.

So since the integral converges, the series also converges.

PS1

#7 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{8n^2-5n}}$

Limit Comparison Test:

$\sum \frac{1}{n^{2/3}}$ is a divergent p-series. Look at:

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt[3]{8n^2-5n}}}{\frac{1}{n^{2/3}}} = \lim_{n\to\infty} \frac{n^{2/3}}{\sqrt[3]{8n^2-5n}} = \lim_{n\to\infty} \frac{\frac{n^{2/3}}{n^{2/3}}}{\sqrt[3]{\frac{8n^2}{n^2} - 5\frac{n}{n^2}}}$$

$$= \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$$

So by the limit comparison test it's "tied" to a divergent series and must diverge.

PS 1

#8 $\sum_{n=1}^{\infty} \sqrt{\frac{n-1}{n}}$

Use Test for Divergence:

$\lim_{n\to\infty} \sqrt{\frac{n-1}{n}} = \lim_{n\to\infty} \sqrt{\frac{1-1/n}{1}} = 1$

and so since the terms don't go to zero, the series diverges.

PS 1

\#9 $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1 \right)$

Use Limit Comparison Test w/ $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n \to \infty} \frac{2^{1/n} - 1}{1/n}$$ both num. and denom. go to 0, so use L'Hopital...

$$= \lim_{n \to \infty} \frac{\ln 2 \cdot 2^{1/n} \cdot \cancel{(-1 n^{-2})}}{\cancel{-1 n^{-2}}} = \ln 2$$

so it's "tied" to something that diverges and thus diverges itself.

PS1

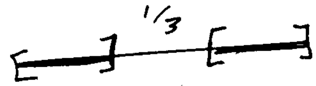
#10 The melon drops h, then $\frac{1}{10}h$, then $\frac{1}{10^2}h$, etc., so

$$h + \frac{1}{10}h + \frac{1}{10^2}h + \dots = \sum_{n=1}^{\infty} h\left(\frac{1}{10}\right)^{n-1} = \frac{h}{1-1/10} = \frac{h}{9/10} = \frac{10}{9}h$$

and in between these drops there are rises of $\frac{1}{10}h$, $\frac{1}{10^2}h$, etc., totalling $\frac{1}{10}$ the drops, so $\frac{1}{9}h$

for a total distance traveled of $\frac{11}{9}h$

PS 1

\#11a First run we remove the middle $\frac{1}{3}$ 
for total of $1/3$

Second we take two pieces of length $\frac{1}{9}$ each 
for total of $2/9$

Third we take four pieces of length $\frac{1}{27}$ each
for total of $\frac{4}{27}$

$\vdots$

etc.

so we're removing $\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\frac{8}{81}+\ldots = \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$

$$= \frac{1/3}{1-2/3}$$

$$= 1$$

Yet in all this lots of points are never removed, for instance $1/3$ and $2/3$, $1/9$ and $2/9$ . . .

In fact it turns out the number of points left after we've removed a total length of one from our original segment of length one is infinite.

PS1

#11b In the first step we remove $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

" " second step we remove eight little squares each $\frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$

so that's $\frac{8}{81}$

" " third step we remove sixty four little squares each $\frac{1}{27} \times \frac{1}{27} = \frac{1}{27^2}$

⋮

So we're taking $\frac{1}{9} + \frac{8}{81} + \frac{64}{729} + \ldots$

$$= \sum_{n=1}^{\infty} \frac{1}{9} \cdot \left(\frac{8}{9}\right)^{n-1}$$

$$= \frac{1/9}{1 - 8/9}$$

$$= 1$$

PS1

#12 The total overhang is given by $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\dots$ which is one-half the harmonic series. Since the harmonic series diverges, this must also, so if we add enough books we can get any overhang we might want.

To see that it doesn't fall, probably the most natural way is to look at stacks of two or three books and see if we can generalize on the pattern.

One book of course has its center of mass (barely) above the table.

Two books gives us a top book centered $1/4$ of a booklength to the right of the table and the second one centered $1/4$ of a booklength left of the edge, balancing perfectly.

Three books gives us the previous two books centered $1/6$ of a booklength right of the edge and the third centered $1/3$ of a booklength left of the edge, and again $2 \cdot 1/6$ balances $1 \cdot 1/3$

Then the pattern extends to n books with $(n-1)$ of them centered $1/2n$ of a book length right and 1 of them centered $\frac{n-1}{2n}$ left of the edge (because it hung off by $\frac{1}{2n}$), so they'll still balance.

Wacky, huh?

PS1

Bonus $1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot \ldots \cdot (2n-1)$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10\cdot 11\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot 8\cdot 10\cdot \ldots}$$

$$= \frac{(2n-1)!}{2^{n-1}(n-1)!}$$