Problem Set #2

\begin{align*}
\text{1.} \quad & \text{Find the 7th degree Maclaurin polynomial for the function } f(x) = \ln(1 + x) \\
& f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \ln(1 + x) \\
& f'(x) = \frac{1}{1 + x} = \frac{1}{1!} x^{-1} \\
& f''(x) = -\frac{1}{(1 + x)^2} = \frac{1}{2!} x^{-2} \\
& f'''(x) = -\frac{2}{(1 + x)^3} = \frac{1}{3!} x^{-3} \\
& f^{(4)}(x) = -\frac{6}{(1 + x)^4} = \frac{1}{4!} x^{-4} \\
& f^{(5)}(x) = -\frac{24}{(1 + x)^5} = \frac{1}{5!} x^{-5} \\
& f^{(6)}(x) = -\frac{120}{(1 + x)^6} = \frac{1}{6!} x^{-6} \\
& f^{(7)}(x) = -\frac{720}{(1 + x)^7} = \frac{1}{7!} x^{-7} \\
& f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = -1, \quad f^{(4)}(0) = -1, \quad f^{(5)}(0) = -1 \\
& f^{(6)}(0) = -1, \quad f^{(7)}(0) = -1 \\
& f(x) = \ln(1 + x) = 0 + \frac{1}{1!} x - \frac{1}{2!} x^2 + \frac{21}{3!} x^3 - \frac{3}{4!} x^4 + \frac{41}{5!} x^5 - \frac{5}{6!} x^6 + \frac{61}{7!} x^7 \\
& = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \\
& \text{b.) Use your approximation from part a to estimate } \ln(1.1) \text{ and } \ln(1.3) \\
& \ln(1.1) = \ln(1 + 0.1) \\
& = 0.1 - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \frac{(0.1)^5}{5} - \frac{(0.1)^6}{6} + \frac{(0.1)^7}{7} \\
& = 0.09531618 \\
& \ln(1.3) = \ln(1 + 2) = 2 - \frac{2^2}{2} + \frac{2^3}{3} - \frac{2^4}{4} + \frac{2^5}{5} - \frac{2^6}{6} + \frac{2^7}{7} = 12.6857 \\
\end{align*}

2. a) Graph the 7th and 8th degree MacLaurin polynomials for } f(x) = \ln(1 + x) \\
\text{together with } f(x) \\
\text{first } f^{(7)}(x) = -7! (1 + x)^{-7}, f^{(8)}(0) = -7! \\
\text{7th degree MacLaurin polynomial } = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} \\
\text{8th degree MacLaurin polynomial } = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \frac{x^8}{8} \\
\text{Graph is on computer paper.}
b.) According to my answers, the approximation for \( \ln(1.1) \) was very accurate and my approximation for \( \ln(3) \) was very inaccurate. This is because for this function, the Maclaurin polynomial works for values of \( x = (-1, 1) \). So I can only get an accurate value for values \( \ln(0 \leq x \leq 2) \), because my function is \( \ln(1+x) \) exactly.

C.) Using a higher degree Maclaurin polynomial would not improve the situation. It would make the approximation for \( \ln(1.1) \) more accurate, but it would make the approximation for \( \ln(3) \) less accurate. To improve the situation, the original function must be changed.
3a) \( x(t) = t^3, \quad y(t) = t^3 - ct \)

When \( c \) is a large negative number the graph stays very close to the \( y \)-axis. It is slightly concave down for \( t, y \) values and slightly concave up for negative \( y \) values. As \( c \) approaches 0 from negative, it looks more like a big "less than" sign with its point at \((0,0)\). As \( c \) is positive and small, it looks like the "less than" sign has been shifted to the right and a small loop sits between \((0,0)\) and the "less than" sign. As \( c \) grows large and positive, the "less than" sign shifts further right and the loop is extended vertically and horizontally.

b) \( \frac{dy}{dx} = \frac{3t^2 - c}{2t} \)

\[ \begin{array}{ll}
\text{horizontal when numerator = 0} & 3t^2 - c = 0 \\
\text{when } t = \pm \sqrt{\frac{c}{3}} & \end{array} \]

\[ \begin{array}{ll}
t = \sqrt{\frac{c}{3}} & y = \left(\frac{c}{3}\right)^{\frac{3}{2}} - c \sqrt{\frac{c}{3}} = \left(\frac{c}{3}, \left(\frac{c}{3}\right)^{\frac{3}{2}} - c \sqrt{\frac{c}{3}}\right) \\
t = -\sqrt{\frac{c}{3}} & y = -\left(\frac{c}{3}\right)^{\frac{3}{2}} + c \sqrt{\frac{c}{3}} = \left(\frac{c}{3}, -\left(\frac{c}{3}\right)^{\frac{3}{2}} + c \sqrt{\frac{c}{3}}\right) \end{array} \]

C) \( x \)-intercepts are when \( y = 0 \):

\[ t^2 - c = 0 \Rightarrow t(t^2 - c) = 0 \Rightarrow t(t - \sqrt{c})(t + \sqrt{c}) = 0 \]

\[ t = 0, \pm \sqrt{c} \]

To find slope of curve plug into \( \frac{dy}{dx} = \frac{3t^2 - c}{2t} \)

\[ \begin{array}{ll}
t = 0 & \frac{dy}{dx} = \frac{-c}{0} \Rightarrow \text{undefined} \\
t = \sqrt{c} & \frac{dy}{dx} = \frac{3\sqrt{c} - c}{2\sqrt{c}} = \frac{-c\sqrt{c}}{c} = -\sqrt{c} \\
t = -\sqrt{c} & \frac{dy}{dx} = \frac{3(-\sqrt{c}) - c}{2(-\sqrt{c})} = \frac{c\sqrt{c}}{c} = \sqrt{c} \end{array} \]
5. \( \text{Area in loop} = \int_{x^0}^{x^h} y(t) \cdot x'(t) \, dt \)

Since it is symmetrical we can use

\[
-2 \int_{0}^{c} (t^2 - ct)(2t) \, dt
\]

\[
= -2 \int_{0}^{c} (2t^3 - 2ct^2) \, dt = -4 \int_{0}^{c} t^3 - ct^2 = -4 \left[ \frac{t^4}{4} - \frac{ct^3}{3} \right]_0^c
\]

\[
= -4 \left[ \frac{c^{7/2}}{5} - \frac{c^{5/2}}{3} \right] = -4 \left[ \frac{3c^{7/2}}{15} - \frac{5c^{5/2}}{15} \right] = \frac{8c^{5/2}}{15}
\]
\[ x = |OQ| - |PQ| = r\theta - d \sin \theta \quad \text{(from \Delta QPC)} \]

\[ y = |QC| - |QC| = r - d \cos \theta \quad \text{(the same \textit{a})} \]

\[ x = r \theta - d \sin \theta \quad y = r - d \cos \theta \]

Sketches:

- For \( d \cdot r \)
- For \( d > r \)
6. Length of curve. \[ x = r \theta - d \sin \theta \]
\[ y = r - d \cos \theta \]

\[ L = \int_{0}^{2\pi} \sqrt{(r^2 + d^2 \sin^2 \theta)^2 + d^2 \cos^2 \theta} \, d\theta \]

\[ L = \int_{0}^{2\pi} \sqrt{r^2 \theta^2 + 2rd \cos \theta} \, d\theta \]

(a) \[ r = 10 \quad d = 5 \]

\[ L = \int_{0}^{2\pi} \sqrt{125 - 100 \cos \theta} = 5 \int_{0}^{2\pi} \sqrt{5 - 10 \cos \theta} \approx 66.824466 \]

(b) \[ r = 10 \quad d = 15 \]

\[ L = \int_{0}^{2\pi} \sqrt{325 - 300 \cos \theta} \approx 105.050227 \]

P.S. It took me a lot of time to try to figure out \[ \int_{0}^{2\pi} \sqrt{5 - 10 \cos \theta} \] without a calculator... 😅