Exam 1 Calc 2 2/26/2003

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. **Integrate** $\int x^3 \ln x \, dx$. [Hint: the answer is $\dfrac{x^4 \ln(x)}{4} - \dfrac{x^4}{16} + C$]

$u = \ln x \quad v = \frac{1}{4}x^4$

$du = \frac{1}{x}\,dx \quad dv = x^3\,dx$

$\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3\,dx$

$= \frac{1}{4}x^4 \ln x - \frac{1}{4}\cdot\frac{1}{4}x^4 + c$

$\int x^3 \ln x \, dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c$

Good

2. If the work required to stretch a spring 1 foot beyond its natural length is 15 ft-lb, how much work is needed to stretch it 6 inches beyond its natural length?

$W(1) = 15 \text{ ft-lbs} = \int_0^1 kx\,dx$

$15 = \frac{k}{2}x^2\Big]_0^1 = \frac{k}{2}$

$30 = k$

Nice Job.

$W(.5) = \int_0^{.5} 30x\,dx$

$= 15x^2\Big]_0^{.5}$

$= \boxed{3.75 \text{ ft-lb}}$

3. Compute the average value of f(x) = sin x on the interval from x = 0 to x = π.

$\text{avg value} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{\int_0^{\pi} \sin x\,dx}{\pi - 0}$

$= \frac{\left[-\cos x\right]_0^{\pi}}{\pi}$

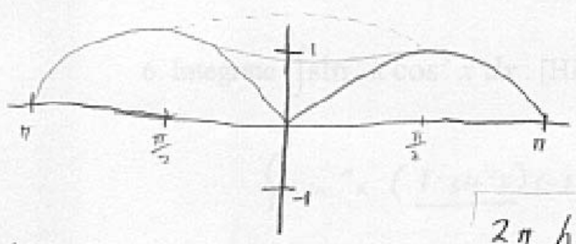
$= \frac{-\cos(\pi) - -\cos(0)}{\pi}$

$= \frac{1 + +1}{\pi} = \frac{2}{\pi}$

Very Nice!

4. **Set up** an integral for the volume of the solid generated by rotating the region between y = sin x, y = 0, x = 0, and x = π around the y axis.

$y = \sin x \quad y = 0 \quad x = 0 \quad x = \pi$



$$\boxed{2\pi \int_0^{\pi} x \sin x \, dx}$$

Good

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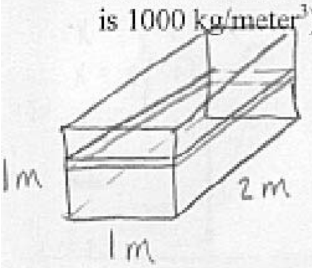
$2\pi \, h \, r$

$2\pi \quad x \quad \sin x$

5. An aquarium 2 meters long, 1 meter wide, and 1 meter deep is full of water. **Set up** an integral for the amount of work needed to pump half the water out of the aquarium (the density of water is 1000 kg/meter^3).



Area of a slice = $2 \ m^2$

Volume of a slice = $2 \Delta y \ m^3$

Force for a slice (vol. x density x accel.) = $2 \Delta y \, m^3 \cdot 1000 \, kg/m^3 \cdot 9.81 \, m/s^2 = 19620 \Delta y$ N

Work for a slice (force x distance travelled) = $19620 \Delta y (1-y)$ N·m

Total work = $\int_{.5}^{1} 19620(1-y) \, dy$

Very well done

6. Integrate $\int \sin^n x \cos^3 x \, dx$. [Hint: the answer is $\dfrac{\sin^{n+1} x}{n+1} - \dfrac{\sin^{n+3}}{n+3} + C$]

Let $u = \sin x$

$du = \cos x \cdot dx$

$I = \int (\sin x)^n \cdot (1 - \sin^2 x) \cos x \cdot dx$

$= \int u^n (1 - u^2) \cdot du$

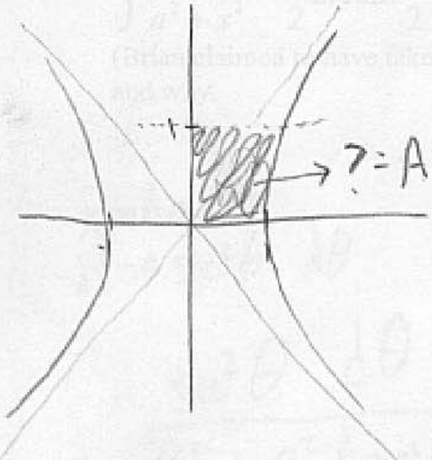
$= \int u^n \, du - \int u^{n+2} \cdot du$

$= \dfrac{u^{n+1}}{n+1} - \dfrac{u^{n+3}}{n+3} + C$

$= \dfrac{\sin^{n+1} x}{n+1} - \dfrac{\sin^{n+3}}{n+3} + C$

Nice Job

7. The graph of $x^2 - y^2 = 1$ is a hyperbola. Set up an integral for the area in the first quadrant bounded by the hyperbola and the line $y = 1$ and use it to find the area.



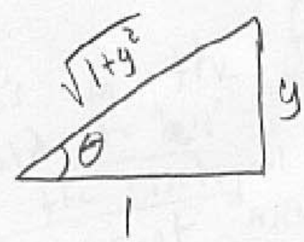
$$x^2 - y^2 = 1$$
$$x^2 = 1 + y^2$$
$$x = \sqrt{1+y^2}$$

$5 + c = 1$

$t + 1 = 5$

$$A = \int_0^1 \sqrt{1+y^2}\, dy$$

$$y = \tan\theta$$
$$dy = \sec^2\theta \, d\theta$$



$$A = \int_0^1 \sqrt{1+\tan^2\theta}\ \sec^2\theta \, d\theta$$

$$A = \int_0^1 \sqrt{\sec^2\theta}\ \sec^2\theta \, d\theta$$

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$$A = \int_0^1 \sec^3\theta \, d\theta$$

$$A = \left[\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|\right]_0^1$$

Well done

$$A = \left[\frac{1}{2}\sqrt{1+y^2}\,(y) + \frac{1}{2}\ln\left|\sqrt{1+y^2} + y\right|\right]_0^1$$

$$A = \left[\frac{1}{2}(\sqrt{2})(1) + \frac{1}{2}\ln\left|\sqrt{2}+1\right|\right] - (0) = \boxed{\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)} \approx \boxed{1.14779 \text{ units}^2}$$

8. Because of a stunningly negligent editor named Brian, the first printing of the second edition of CliffsQuickReview Calculus was released with a table of integrals that said $\int \frac{dx}{a^2+x^2} = \frac{1}{2}\arctan\frac{x}{2} + C$. Explain, in simple enough terms that Brian can follow along (Brian claimed to have taken calculus himself), exactly how this formula is or is not acceptable and why.

Brian,

When editing these integrals it would probably be a good idea to take the derivative of the result, just in case an integration mistake has been made.

$$\frac{d}{dx}\left(\arctan\frac{x}{a}+C\right) = \frac{1}{a^2+x^2} + 0$$

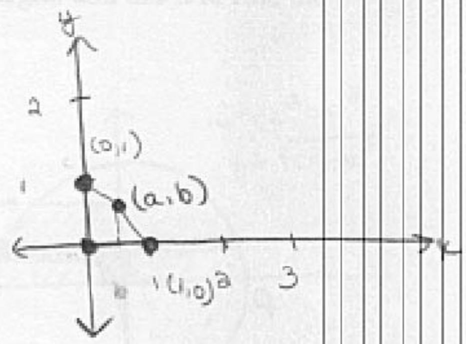
$$\frac{d}{dx}\left(\frac{1}{2}\arctan\frac{x}{2}+C\right) = \frac{1}{4+x^2} + 0$$

YES!

The variable "a" still needs to be represented in the answer as opposed to the number 2.

Excellent.

9. Suppose a quadrilateral has vertices at (0,0), (1,0), (0,1), and a point (a,b) for which $0 < a < 1$ and $b>0$. **Set up** an integral (or integrals) for the area of this quadrilateral.



$$m = \frac{1-b}{0-a} = \frac{1-b}{-a}$$

$$y - 1 = \frac{1-b}{-a}(x - 0)$$

$$y = \frac{x(1-b)}{-a} + 1$$ ← equation for line between (0,1) & (a,b)

$$m = \frac{b-0}{a-1} = \frac{b}{a-1}$$

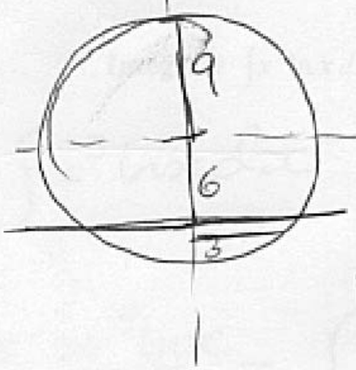
$$y - 0 = \frac{b}{a-1}(x-1)$$

$$y = \frac{bx}{a-1} - \frac{b}{a-1}$$ ← equation " " (a,b) & (1,0)

Nicely Done!

$$A = \int_0^a \left[\frac{x(1-b)}{-a} + 1\right] dx + \int_a^1 \left[\frac{bx}{a-1} - \frac{b}{a-1}\right] dx$$

10. Jon has a bowl-shaped fountain on his desk which is shaped like a frustum of a sphere with a radius of 9 inches, cut off 3 inches up from the bottom. Set up an integral and use it to find the volume of water contained in this fountain.



$y^2 + x^2 = 9^2 \qquad \pi r^2 dr$

$x = \sqrt{9^2 - y^2} \qquad r = x$

$V = \pi \int_{6}^{9} \left(\sqrt{9^2 - y^2}\right)^2 dy$

$V = \pi \int_{-9}^{-6} 9^2 - y^2 \, dy$

$V = \pi \left[81y - \frac{y^3}{3} \right]_{6}^{9}$

$= \pi \left[81(9) - \frac{9^3}{3} - 81(6) + \frac{6^3}{3} \right]$

$\boxed{V = 72\pi \text{ inch}^3}$

Excellent.