Exam 3 Calc 2 4/18/2003

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Convert the point with polar coordinates $r = 4$, $\theta = 5\pi/6$ to rectangular coordinates.

$r = 4$

$\theta = 5\pi/6$

Good

conversion formulas:

$x = r\cos\theta$

$y = r\sin\theta$

$x = 4\cos(5\pi/6)$

$x = 4 \cdot -\frac{\sqrt{3}}{2}$

$x = -2\sqrt{3}$

$y = 4\sin(5\pi/6)$

$y = 4 \cdot \frac{1}{2}$

$y = 2$

$(-2\sqrt{3}, 2)$

2. Determine whether $y = e^{2t}$ is a solution to the differential equation $y'' + 3y' - 10y = 0$.

$y = e^{2t}$

$y' = 2e^{2t}$

$y'' = 4e^{2t}$

$y'' + 3y' - 10y = 0$

$4e^{2t} + 3(2e^{2t}) - 10(e^{2t}) = 0$

$4e^{2t} + 6e^{2t} - 10e^{2t} = 0$

$10e^{2t} - 10e^{2t} = 0$

$0 = 0$ ✓

Well done

yes, $y = e^{2t}$ is a solution.

3. If a hot cup of coffee is left in a 70° F room, it is found that a general solution to the corresponding differential equation is $T(t) = Ke^{-t/30} + 70$. **Find a particular solution** representing a cup which begins at 190° F.

$T(t) = Ae^{-t/30} + 70$ find A using the initial condition $y(0)=190$

$190 = Ae^{0} + 70$

$A = 120$

The particular soln. for these conditions is $T(t) = 120e^{-t/30} + 70$

Great!

4. **Set up an integral** for the length of the curve with parametric equations

$x = 2\sin 2t \qquad y = 2\sin t$

$\frac{dx}{dt} = 4\cos 2t \qquad \frac{dy}{dt} = 2\cos t$

between the point (0, 0) and the point $\left(\sqrt{3}, \sqrt{3}\right)$.

Arc length parametric curve $= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Arc length $= \int_0^{\pi/3} \sqrt{(4\cos 2t)^2 + (2\cos t)^2}\, dt$

Well done!

to find the limits of integration in terms of t

$0 = 2\sin 2t$

$t = \frac{\sin^{-1} 0}{2}$

$t = \frac{0}{2}, \frac{\pi}{2}$ or $\frac{2\pi}{2}$

$t = \boxed{0}, \pi$

$0 = 2\sin t$

$t = \sin^{-1} 0$

$t = \boxed{0}, \pi, 2\pi$

$\sqrt{3} = 2\sin 2t$

$\sqrt{3}/2 = \sin 2t$

$2t = \sin^{-1}(\sqrt{3}/2)$

$t = \frac{\sin^{-1}(\sqrt{3}/2)}{2}$

$t = \pi/6$

$\sqrt{3} = 2\sin t$

$\sqrt{3}/2 = \sin t$

$t = \sin^{-1}(\sqrt{3}/2)$

$t = \pi/3, 2\pi/3$

5. Let $C(t)$ be the concentration of a substance in a person's blood. If the body eliminates the substance at a rate proportional to the current concentration, then the concentration will follow the differential equation $\dfrac{dC}{dt} = -kC(t)$. **Find a general solution** to this differential equation.

$$\frac{dC}{dt} = -kC(t)$$

$$\int \frac{dC}{C(t)} = \int -k\,dt$$

absolute value bars can be removed because you can't have a negative concentration :)

$$\ln|C(t)| = -kt + C$$

$$C(t) = e^{-kt+C}$$

$$\boxed{C(t) = Ae^{-kt}}$$

6. Find the exact coordinates of the lowest point on the curve with parametric equations

$$x = 3t^3 + t, \; y = 2t^2 + t.$$

$x = 3t^3 + t$

$\Rightarrow \frac{dx}{dt} = 9t^2 + 1$

$y = 2t^2 + t$

$\Rightarrow \frac{dy}{dt} = 4t + 1$

$\therefore \frac{dy}{dx} = \frac{4t+1}{9t^2+1}$

The ~~eq~~ lowest point on the curve will ~~be~~ exist where the tangent has a slope $= 0$.

i.e. $\frac{dy}{dx} = 0$

$\Rightarrow \frac{4t+1}{9t^2+1} = 0$

$\Rightarrow 4t + 1 = 0$

$\Rightarrow 4t = -1$

$\therefore t = \frac{-1}{4}$

Putting t's value in x and y

$\Rightarrow x = 3\left(-\frac{1}{4}\right)^3 + \left(-\frac{1}{4}\right) = -\left[\frac{3}{64} + \frac{1}{4}\right] = -\left[\frac{3+16}{64}\right]$

$= \frac{-19}{64}$

$\Rightarrow y = 2\left(-\frac{1}{4}\right)^2 + \left(-\frac{1}{4}\right) = \frac{2}{16} - \frac{1}{4} = \frac{1-2}{8}$

$= -\frac{1}{8}$ $\quad \therefore$ Coordinates $= \left(-\frac{19}{64}, -\frac{1}{8}\right)$

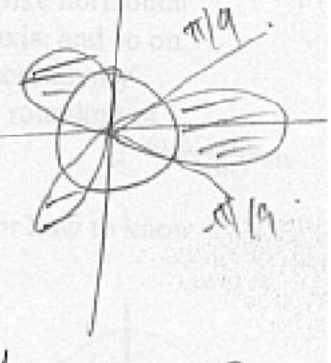
7. **Set up an integral** (or integrals) for the arc length of the portion(s) of the curve $r = \cos 3\theta$ which lies outside the curve $r = ½$.

point of intersection

$\frac{1}{2} = \cos 3\theta$

$3\theta = \frac{\pi}{3}$

$\theta = \frac{\pi}{9}$



$r = \cos 3\theta$

$\frac{dr}{d\theta} = -3\sin 3\theta$

$$\text{Arc Length} = 3\int_{-\frac{\pi}{9}}^{\frac{\pi}{9}} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

$$= 6\int_{0}^{\frac{\pi}{9}} \sqrt{(9\sin^2 3\theta) + \cos^2 3\theta}\, d\theta$$

Nice Job!

8. Beth is a calculus student at E.S.U. who wants some help with polar equations. Beth says "So I was talking about our homework assignment with this guy in my class, and there was this question about when you've got the polar equation $r = \theta$. So it's this spiral that goes around and around, and he says that it's horizontal and vertical when it crosses the axes, so like horizontal when it crosses the positive y axis, then vertical when it crosses the negative x axis, and so on. I told him it doesn't really look quite like those are the horizontal and vertical places on my calculator, but he said the calculator just distorts it a little because of the dots or rounding or something. I'm pretty sure I'm right, but I have no idea how to prove it."

Explain to Beth, in terms she can understand, either how to be sure she's right, or how to know her classmate is right.

By finding the first derivative of the equation and evaluating it at points where the curve crosses the x or y axis you can tell if it is horizontal or vertical. Yes!

$r=\theta \qquad x = r\cos\theta = \theta\cos\theta \qquad \frac{dx}{d\theta} = -\theta\sin\theta + \cos\theta$

$\qquad\qquad y = r\sin\theta = \theta\sin\theta \qquad \frac{dy}{d\theta} = \theta\cos\theta + \sin\theta$

$$\frac{dy}{dx} = \frac{\theta\cos\theta + \sin\theta}{-\theta\sin\theta + \cos\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{\frac{\pi}{2}(0)+1}{-(\frac{\pi}{2})(1)+0} = \frac{1}{-\frac{\pi}{2}} = \frac{-2}{\pi}$$ the slope at $\frac{\pi}{2} = \frac{-2}{\pi}$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi} = \frac{\pi(-1)+0}{-\pi(0)+1} = -\pi$$ the slope at $\pi = -\pi$

$\frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}$

9. **Set up an integral** (or integrals) for the area of the region between the inner and outer loops of the curve r = 2 + 4sin θ.

$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$

$2 + 4\sin\theta = 0$

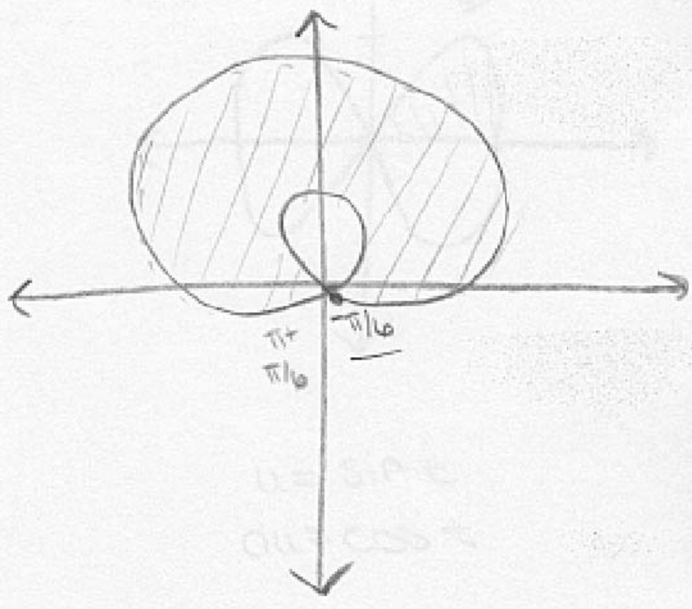
$4\sin\theta = -2$

$\sin\theta = \frac{-1}{2}$

$\theta = \sin^{-1}(-1/2)$

$A = \int_{-\pi/6}^{7\pi/6} \frac{1}{2}(2+4\sin\theta)^2 \, d\theta - \int_{7\pi/6}^{11\pi/6} \frac{1}{2}(2+4\sin\theta)^2 \, d\theta$

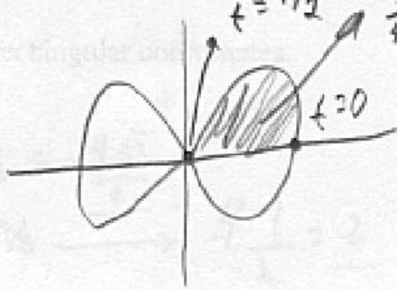
Excellent



10. **Find the area** bounded by the parametric curve with equations x = cos t, y = sin 2t [Hint: Depending on how you proceed, you might find the trig identity sin 2x = 2sin x cos x useful].

$\frac{1}{4}$ of total A

$$A = \int_A^B y(t)\, x'(t)\, dt$$



$$A = -4\underline{\int_0^{\pi/2} (\sin 2t)(-\sin t)}$$

b/c limits go from $\underline{R \text{ to } L}$, instead of L → R, like normal, Yes!

$u = \sin t$

$s + c = 1$

$s = 1 - c$

$$A = -(-4)\int_0^{\pi/2} \sin 2t \sin t \, dt$$

$u = \sin t$

$du = \cancel{\cos t}\, dt$

$\frac{du}{\cos t}$

$$A = 4\int_0^{\pi/2} \underline{2 \sin t \cos t \sin t} \, dt$$

Nice

$$A = 8\int_0^{\pi/2} \underline{\sin^2 t \cos t} = 8\int_0^{\pi/2} \underline{\frac{u^2 \cancel{\cos t}}{\cancel{\cos t}}} \, du$$

$$\cancel{8\int_0^{\pi/2} \cos t} \; 8\int_0^{\pi/2} u^2 \, du = \frac{8}{3}\left[u^3\right] = \frac{8}{3}\left[\sin t\right]^3 \Big|_0^{\pi/2}$$

$$= \frac{8}{3}(1) - \frac{8}{3}(0) = \boxed{\frac{8}{3}}$$