Final Exam A Calculus 2 5/21/2003

Each problem is worth 10 points, show all work and give adequate explanations for full credit. Please keep your work as legible as possible.

Warning: This is a practice test, not a study guide. If all you know is how to do the problems on this test, you'll probably fail the real final, because the selection of topics there may be significantly different than what's represented here. Still, a warm-up for the exam might be a good thing if used properly, so have fun with it.

Bear in mind that only answers are given here, and that without justification these would generally receive at most half credit.

1. Find the slope of the line tangent to the curve with parametric equations $x = 3t^3$, $y = 4t^2 - 2$ at the point (-3,2).

-8/9

2. Write an integral for the area inside the curve with polar coordinates $r = 1 + \cos\theta$ and to the right of the y axis.

One possibility would be:

$$\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos x)^2$$

3. Set up an integral for the area below the curve $y = xe^{-x}$ and above the positive x axis and evaluate it.

$$\int_0^{\infty} xe^{-x}\,dx = \mathbf{1}$$

4. Set up an integral and use it to compute [added 5/20] the average value of the function $f(x) = \sin^3 x \cos^2 x$ on the interval from $x = 0$ to $x = \pi$.

$$\frac{1}{\pi - 0}\int_0^{\pi} \sin^3 x \cos^2 x\,dx = \frac{4}{15\pi}$$

5. After a long and drawn-out process involving a couple of pages of work, a differential equations student determines that $\sum_{n=0}^{\infty} \frac{n}{3^n}(x-4)^n$ is a Taylor series for the solution to a differential equation. Unfortunately, they've completely forgotten everything they ever learned about interval of convergence. Find the interval of convergence of their series, and make sure your work is coherent enough that they can understand what you're doing.

(1,7)

6. Set up an integral and evaluate it to find the amount of work done in pumping the water out of a half-full spherical tank with a radius of 3 meters to a point 1 meter above the top of the tank.

There are several valid possible arrangements depending on choices of axes and such, but one possibility is $9800\pi\int_0^3 (9-x^2)(4+x)\,dx = 904050\pi$ Joules. The answer should be the same regardless of which way you set it up.

7. Set up an integral and use it to find the volume of the solid obtained by revolving the region between the curves $y = x^2$ and $y = x + 2$ around the line $y = -2$.

One possible setup is $\pi\int_{-1}^{2}\left\{\left[\left(x+1\right)+2\right]^2-\left[\left(x^2\right)+2\right]^2\right\}dx=\frac{42\pi}{5}$.

8. A calculus student has it written in his notes that the ratio test indicates convergence if its results are less than or equal to 1, but got some things marked wrong on his test when he used that. Explain clearly to the student how he should have known that a 1 from the ratio test doesn't necessarily indicate convergence.

There are lots of ways to approach this, but some very good options would involve pointing out that lots of series we definitely know diverge, like the harmonic series or the series 1 + 1 + 1 + ..., will give a result of 1 in the ratio test, so a 1 from the ratio test couldn't possibly assure convergence.

9. Jon apparently has root grubs on the grass in his yard. If the root grub population grows logistically according to the equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{K}\right)$$

where y(t) is the number of grubs at time t (measured in months after the last frost), y(0) = 100, the carrying capacity is estimated to be 12,000 grubs, and k = 0.12, find a solution (showing how you worked it out) to the differential equation and use it to predict the root grub population by fall, 6 months after the last frost.

The solution will be something along the lines of $y = \dfrac{12000}{119e^{-0.12t} + 1}$, and the prediction for y(6) is about 204 grubs.

10. Find the value of b for which the length of the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}$ on the interval $[2,b]$ is $6 + \frac{\ln 2}{4}$.

$b = 4$