Exam 2 Differential Equations 4/11/2003

Each problem is worth 10 points. For full credit provide complete justification for your answers.

Use the following system of differential equations for problems 1 and 2:

$$\frac{dx}{dt} = y + y^2$$

$$\frac{dy}{dt} = -\frac{x}{2} + \frac{y}{5} - xy + \frac{6y^2}{5}$$

1. Verify that (0,0) is an equilibrium point of the system.

$\frac{dx}{dt} = 0 + 0^2$ $\frac{dx}{dt} = 0$

$\frac{dy}{dt} = -\frac{0}{2} + \frac{0}{5} - 0(0) + \frac{6(0)^2}{5}$ $\frac{dy}{dt} = 0$

Excellent

When the derivatives are equal to zero, the system is in equilibrium. Therefore, since I got $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$, (0,0) is an equilibrium point.

2. Find the other equilibrium point of the system.

$\frac{dx}{dt} = 0$ $\frac{dy}{dt} = 0$

$0 = y + y^2$

$0 = y(1+y)$

$y = 0$ or $y = -1$

we know when $y = 0$, $x = 0$, so we plug $y = -1$ into the other differential equation.

$0 = -\frac{x}{2} + \frac{y}{5} - xy + \frac{6y^2}{5}$

$0 = -\frac{x}{2} - \frac{1}{5} - x(-1) + \frac{6(-1)^2}{5}$

$0 = -\frac{x}{2} - \frac{1}{5} + x + \frac{6}{5}$

$0 = \frac{1}{2}x + 1$

$-1 = \frac{1}{2}x$

$x = -2$

Great

The other equilibrium point is $\boxed{(-2,-1)}$

3. Give an example of a partially decoupled system of differential equations.

$$\frac{dx}{dt} = x$$

$$\frac{dy}{dt} = xy$$

Excellent

1 equation depends only on its dependent variable

the other depends on both dependent variables

4. Determine whether $(4e^{2t} - e^{-t}, 3e^{-t})$ is a solution to the system of differential equations

$$\frac{dx}{dt} = 2x + y$$

$$\frac{dy}{dt} = -y.$$

$x = 4e^{2t} - e^{-t}$ $\qquad$ $y = 3e^{-t}$

$\frac{dx}{dt} = 8e^{2t} + e^{-t}$ $\qquad$ $\frac{dy}{dt} = -3e^{-t}$

$8e^{2t} + e^{-t} \overset{?}{=} 2(4e^{2t} - e^{-t}) + 3e^{-t}$ $\qquad$ $-3e^{-t} \overset{?}{=} -(3e^{-t})$ ✓ yes

$= 8e^{2t} - 2e^{-t} + 3e^{-t}$

✓ $= 8e^{2t} + e^{-t}$

yes

Great

$\boxed{\text{yes } (4e^{2t} - e^{-t}, 3e^{-t}) \text{ is a solution}}$

For questions 5 through 7, consider a situation where the populations of felines and rodents in a region follow the differential equations

-50+40

$$\frac{dF}{dt} = -5F + 0.01FR$$

1200 - 800

$$\frac{dR}{dt} = 3R - 0.2FR.$$

5. Use Euler's Method with a step size of $\Delta t = 0.25$ to estimate the populations of felines and rodents in the region after 1 month if those populations begin at F = 10 and R = 400. [Keep all of your intermediate results to at least the nearest tenth of a creature, however silly that might be.]

(10,400) + .25(-10,400) = (7.5,500)

(7.5,500) + .25(0,750) = (7.5,687.5) Nice

(7.5,687.5) + .25(14.06,1044.75) = (11.02,948.69)

(11.02,948.69) + .25(49.45,755.16) = (23.38,1137.48)

after 1 month

F = 23.38

R = 1137.48 Great

6. Use Euler's Method with a step size of $\Delta t = 1$ to estimate the populations of felines and rodents in the region after 1 month with the initial condition above. Explain how valid this approximation is likely to be and why.

(10,400) + 1(-10,400) = (0,800)

after 1 month

F = 0

R = 800

Absolutely so.

not valid approximation. step size is too big so the change is more dramatic than in real life. Also it doesn't allow for what happens in between 0 months and 1 month. The smaller the step size, the more accurate the approximation.

7. Explain what the positive and negative signs on the FR terms in the system of differential equations on the previous page mean, that is, what do they tell you about the situation being represented?

In the $\frac{df}{dt}$ equation, the postive sign of the FR term means that the felines eat rodents, so when felines and rodents interact, the felines eat the rodents and if this term is large enough, the feline population will increase.

In the $\frac{dR}{dt}$ equation, the negative sign on the FR term means that the rodents are being eaten by felines, so when rodents and felines interact, the rodents are eaten and if this term is negative enough, it will cause the rodent population to decrease.

FR interactions have a positive effect on the feline population and a negative effect on the rodent population.

Excellent

8. Suppose the functions $y = e^{2t}\cos(10t)$, $y = e^{-0.2t}\cos(10t)$, $y = e^{-2t}\sin(10t)$, and $y = e^{0.2t}\sin(10t)$ are proposed as possible solutions to a system of differential equations representing the position of a spring acting in a resisting medium. Assess each of these candidate solutions – which of these are particularly reasonable or unreasonable possibilities, and why?

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$y = e^{2t}\cos(10)t$ does not work, because e^{2t} increases as time goes on, needs to make it decrease

$y = e^{-0.2t}\cos(10)t$ might work because the $e^{-0.2t}$ makes it decrease, it would be damped,

$y = e^{-2t}\sin(10t)$ might work as well, e^{-2t} also causes damping.

$y = e^{0.2t}\sin(t)$ does not work, the $e^{0.2t}$ will make y increase as time goes on.

Great

9. Consider the differential equation $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 2x$. Suppose someone suggests that a solution of the form $y = Ax^2 + Bx + C$ exists, for some values of A, B, and C. Find appropriate values of A, B, and C, if there are any.

$$y' = 2Ax + B$$

$$y'' = 2A$$

Substitute:

$$(2A) + 3(2Ax+B) - (Ax^2+Bx+C) = 2x$$

Matching x^2:

$$-A = 0$$

Matching x:

$$\cancel{6}Ax - Bx = 2x$$

$$-B = 2$$

$$B = -2$$

Constants:

$$2A + 3B - C = 0$$

$$0 + 3(-2) - C = 0$$

$$C = -6$$

So $y = 0x^2 - 2x - 6$

Check:

$$0 + 3(-2) - (-2x-6) \stackrel{?}{=} 2x \checkmark$$

10. Solve the system of differential equations

$$\frac{dx}{dt} = -\frac{1}{2}x + 3y$$

$$\frac{dy}{dt} = x.$$

$$\frac{d}{dt}\left(\frac{dy}{dt} = x\right)$$

Good!

$$\frac{d^2y}{dt^2} = \frac{dx}{dt}$$

$$\frac{d^2y}{dt^2} = -\frac{1}{2}\frac{dy}{dt} + 3y$$

guess $y = e^{rt}$

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

$$r^2 e^{rt} + \frac{1}{2} re^{rt} - 3e^{rt} = 0$$

$$e^{rt}\left(r^2 + \frac{1}{2}r - 3\right) = 0$$

$$e^{rt} \neq 0$$

$$r^2 + \frac{1}{2}r - 3 = 0$$

$$(r+2)(r - 3/2) = 0$$

10/10

$r = -2 \quad r = 3/2 \qquad y = e^{-2t}$ or $y = e^{3/2 t}$

Well done!

$$\frac{dy}{dt} = x \qquad \frac{de^{-2t}}{dt} = -2e^{-2t}$$

$$\frac{de^{3/2 t}}{dt} = +\frac{3}{2}e^{3/2 t}$$

$$\boxed{\left(-2e^{-2t}, e^{-2t}\right) \ \& \ \left(\tfrac{3}{2}e^{3/2 t}, e^{3/2 t}\right)}$$