Exam 3 Differential Equations 5/7/2003

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. If you know that $\lambda_1 = -3$ and $\lambda_2 = -2$ are eigenvalues of the coefficient matrix of a planar linear system, and that $\mathbf{V}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{V}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ are the corresponding eigenvectors, write a **general solution** to the system.

$$\boxed{Y(t) = k_1 e^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} -1 \\ 3 \end{pmatrix}}$$

Yep

2. Give an **example** of a coefficient matrix for a planar linear system in which the origin would be classified as a saddle.

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\lambda = -1,\ 3$$

one λ is positive and one is negative

$\lambda - 3 \rightarrow 3 - \lambda$ $\quad\quad \lambda + 1 = -1 - \lambda$

$$A = \boxed{\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}}$$

Nice

3. Give an **example** of a system of two linear differential equations (with real coefficients) whose eigenvalues will be complex.

For that to happen, discriminant (b^2-4ac) must be < 0

So, we ought to make a or c bigger compared to b^2 and their product $-$ve.

So, $A = \begin{pmatrix} -1 & 40 \\ -1 & -1 \end{pmatrix}$ ought to be such a system

Hard way, but totally valid.

or, $\frac{dx}{dt} = -1x + 40y$ $\boxed{\text{Ans}}$.

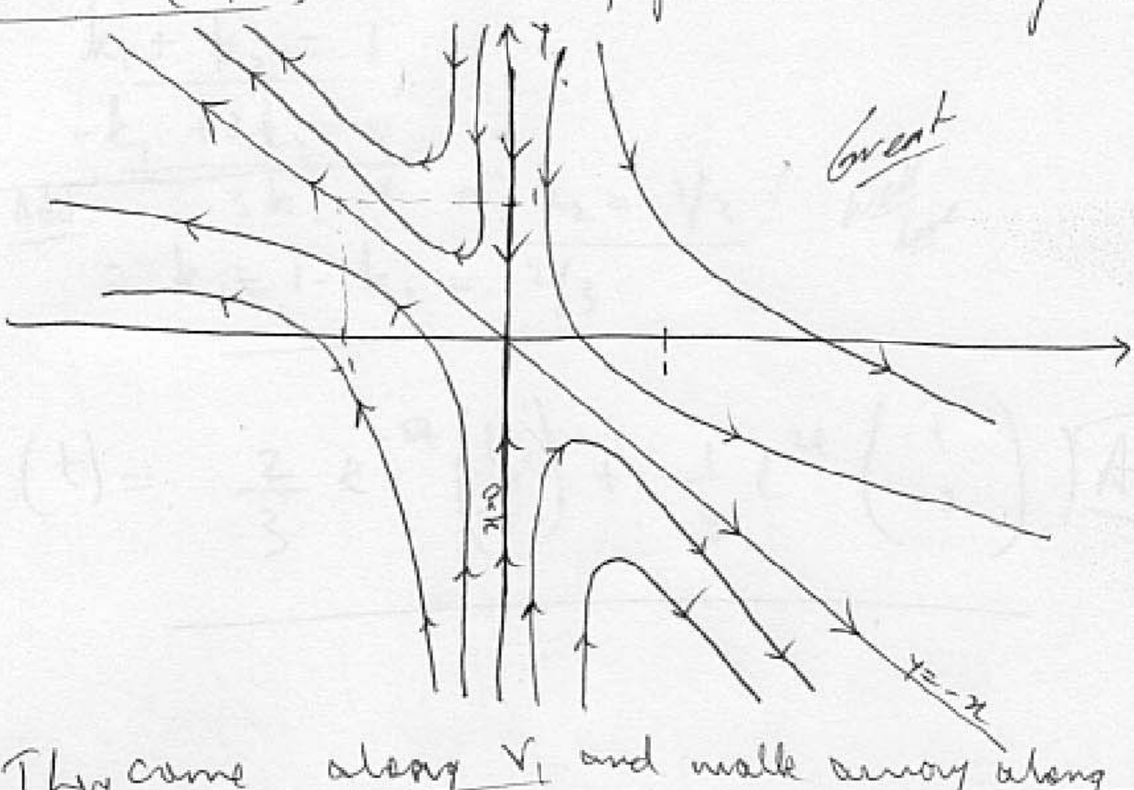
and $\frac{dy}{dt} = -1x - 1y$

Check :- $\begin{vmatrix} -1-\lambda & 40 \\ -1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)^2 + 40 = 0 \Rightarrow \lambda^2 + 2\lambda + 41 = 0$

and $2^2 - 4 \cdot 41 \cdot 1 < 0$. true!

4. **Sketch the phase portrait** for a system of planar differential equations where $\lambda_1 = -2$ and $\lambda_2 = 1$, with $\mathbf{V}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{V}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ being the corresponding eigenvectors.

So, we have a saddle! Along V_1, curves move towards $(0,0)$ and away from $(0,0)$ along V_2. Exactly.



They come along V_1 and walk away along V_2.

5. Given that the planar system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix} \mathbf{Y}$$

has general solution

$$\mathbf{Y}(t) = k_1 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

find the particular solution satisfying $\mathbf{Y}_0 = (1,0)$.

$$\vec{Y}(0) = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} k_1 \\ -k_1 \end{pmatrix} + \begin{pmatrix} k_2 \\ 2k_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$k_1 + k_2 = 1$; $k_1 = 1 - k_2$

$-k_1 + 2k_2 = 0$

$-(1 - k_2) + 2k_2 = 0$

$-1 + k_2 + 2k_2 = 0$

$3k_2 = 1$

$\underline{k_2 = 1/3}$; $\underline{k_1 = 1 - 1/3 = \frac{2}{3}}$

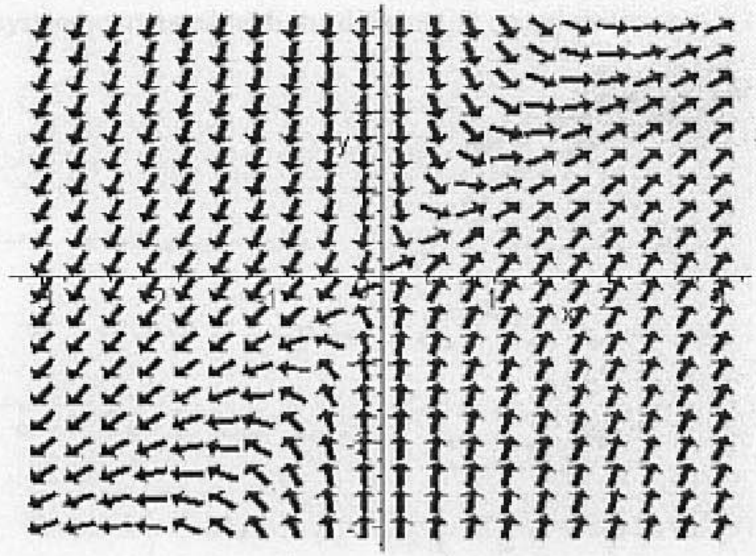
Well done

$$\boxed{\vec{Y}(t) = \frac{2}{3} e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{3} e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

6. **Give a general solution** to the system of differential equations

$$\frac{dx}{dt} = 2x + 0y$$

$$\frac{dy}{dt} = 4x - 3y$$



$A = \begin{pmatrix} 2 & 0 \\ 4 & -3 \end{pmatrix}$ $\quad Y = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \begin{pmatrix} 2 & 0 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

$\det(A - \lambda I) = 0$

$\det \begin{pmatrix} 2-\lambda & 0 \\ 4 & -3-\lambda \end{pmatrix} = 0$

$(2-\lambda)(-3-\lambda) - 0 = 0$

$-6 - 2\lambda + 3\lambda + \lambda^2 = 0$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda + 3)(\lambda - 2) = 0$

$\lambda = 2, -3$

Excellent

For $\lambda = 2$:

$2x = 2x$

$4x - 3y = 2y \rightarrow 4x = 5y$

$x = \frac{5}{4} y$ $\quad \begin{pmatrix} \frac{5}{4} \\ 1 \end{pmatrix}$

For $\lambda = -3$

$2x = -3x$

$4x - 3y = -3y \rightarrow 4x = 0$

$x = 0$ $\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\boxed{Y(t) = k_1 e^{2t}\begin{pmatrix} \frac{5}{4} \\ 1 \end{pmatrix} + k_2 e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$ or

$$Y(t) = k_1 e^{2t}\begin{pmatrix} 5 \\ 4 \end{pmatrix} + k_2 e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

7. **Give a condition** under which the planar system associated with the differential equation

$$ay'' + by' + cy = 0$$

will have purely imaginary eigenvalues.

$e^{st}(as^2 + bs + c) = 0$

so

$as^2 + bs + c = 0$

$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ purely imaginary when $b = 0$

so

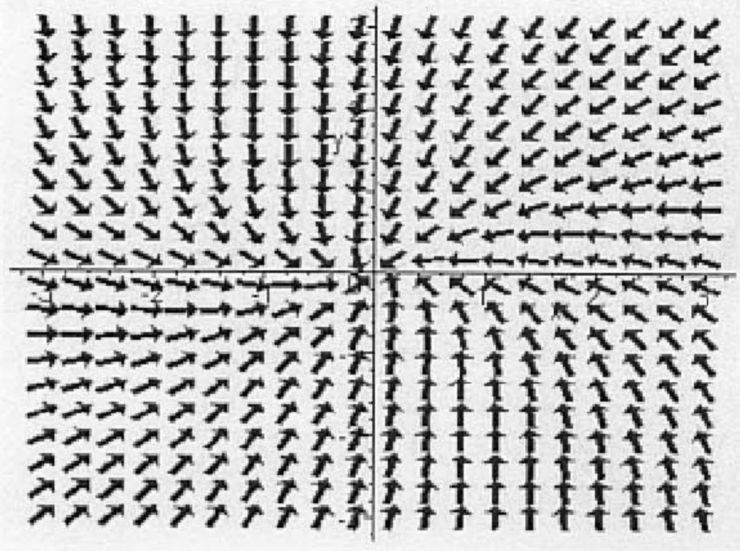
$\dfrac{\pm\sqrt{-4ac}}{2a}$ and a & c are both + or −

Nice

8. For the planar system

$$\frac{dY}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \mathbf{Y},$$

Find the particular solution satisfying the initial condition $\mathbf{Y}_0 = (0,1)$.



$$\vec{A} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$$

$$\det \begin{pmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{pmatrix} = 0 \quad \therefore$$

$$(2+\lambda)(4+\lambda) + 1 = 0$$

$$8 + 4\lambda + 2\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0$$

repeated eigenvalue $\lambda = -3$

$$\vec{V}_1 = (\vec{A} - \lambda \vec{I})\vec{V}_0 \longrightarrow \left(\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \right) \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{V}_0 = (x_0, y_0) = (0, 1)$$

Well Done

$$\left(\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \right) \times \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\vec{Y}(t) = e^{\lambda t}\vec{V}_0 + t e^{\lambda t}\vec{V}_1$$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\boxed{\vec{Y}(t) = e^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t e^{-3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}}$$

$$(2\times 2) \times (2\times 1) = (2\times 1)$$

9. **Show** that a system of the form

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = -bx + ay$$

with $b \neq 0$ must have complex eigenvalues.

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad d\bar{Y} = A\bar{Y}$$

$$\det \begin{pmatrix} a-\lambda & b \\ -b & a-\lambda \end{pmatrix} = 0$$

$$(a-\lambda)(a-\lambda) + b^2 = 0$$

$$a^2 - 2a\lambda + \lambda^2 + b^2 = 0$$

$$\lambda^2 - 2a\lambda + (a^2+b^2) = 0$$

$\rightarrow$ $a = 1$, $b = 2a$, $c = a^2+b^2$ for quadratic equation.

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} \rightarrow \frac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2} \rightarrow \frac{2a \pm \sqrt{4a^2 - 4a^2 - 4b^2}}{2}$$

$$\frac{2a \pm \sqrt{-4b^2}}{2} \rightarrow \frac{2a \pm 2\sqrt{-b^2}}{2} \rightarrow a \pm \sqrt{-b^2}$$

b^2 itself will always be positive so $-b^2$ will always be negative. Therefore since the discriminant is less than zero, the system must have complex eigenvalues.

Outstanding!

10. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Define the trace of **A** to be tr(**A**) = $a + d$. **Show** that A has only one eigenvalue if and only if $(\mathrm{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A}) = 0$.

$(a-\lambda)(d-\lambda) - bc = 0$

$ad - \lambda d - \lambda a + \lambda^2 - bc = 0$

$\lambda^2 - (a+d)\lambda + ad - bc = 0$

$\lambda^2 - (a+d)\lambda + ad - bc$

$$\frac{a+d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

has 1 eigenvalue iff $\sqrt{(a+d)^2 - 4(ad-bc)} = 0$ since $\frac{(a+d) \pm 0}{2}$ gives only 1 value

$(a+d) = tr(A)$ +

$det(A) = ad - bc$ so

$(a+d)^2 - 4(ad-bc) = tr(A)^2 - 4det(A)$

$\therefore$

$tr(A)^2 - 4det(A) = 0$ as well

Nice Job.

$\Rightarrow$ A can only have 1 eigenvalue namely $\lambda = \frac{a+d}{2}$