Practice Exam 1 Calc 2 2/22/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers. Warning: This is only a practice exam. Knowing how to do the problems on this exam is good, but if that's **all** you know, you're not likely to do well at all on the actual exam. Use this as a warm-up, a trial run, or just to get used to the form questions might take, but don't think of it as a complete study guide. Also notice that, since this is basically stitched together from old exams I've given, it doesn't hit quite all the topics our exam will. You might look at problems 28 in section 7.4 and 16 in section 7.5 as a supplement to the material on this exam.

1. Integrate $\int \frac{5}{(2x)^2+9}\,dx$ (feel free to use a table of integrals).

Let $u = 2x$

$\frac{du}{dx} = 2$

$\frac{du}{2} = dx$

$$= \int \frac{5}{u^2+9} \cdot \frac{du}{2}$$

$$= \frac{5}{2}\int \frac{1}{u^2+3^2}\,du \quad \text{Use line 24}$$

$$= \frac{5}{2}\cdot\frac{1}{3}\arctan\frac{u}{3} + C$$

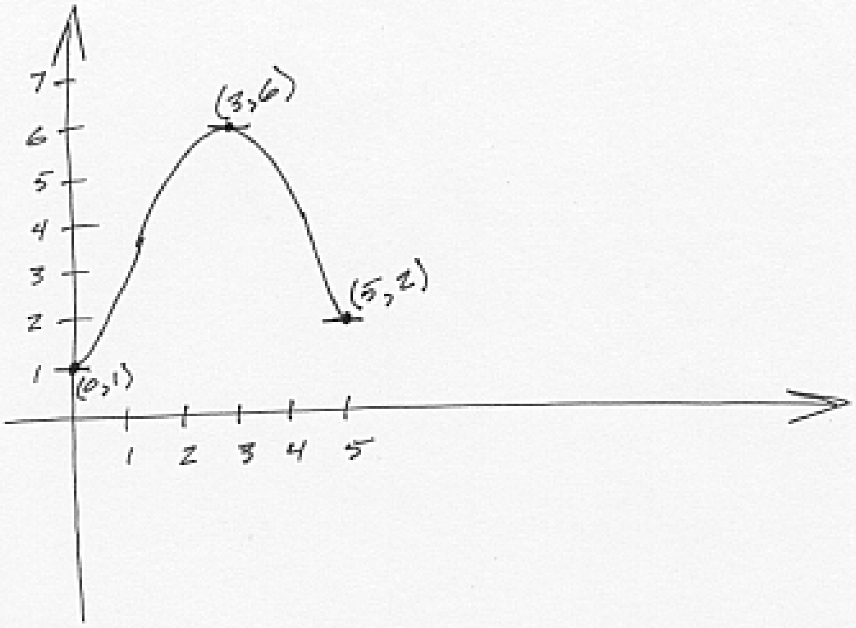
$$= \frac{5}{6}\arctan\frac{2x}{3} + C$$

2. Sketch a graph of the antiderivative of F′(x) if you know:

(1) $F(0) = 1,\ F'(0) = F'(3) = F'(5) = 0$

(2) $\int_0^3 F'(x)\,dx = 5$

(3) $\int_3^5 F'(x)\,dx = -4$



3. **Integrate** $\int x^3 \ln x \, dx$. [Hint: the answer is $\dfrac{x^4 \ln(x)}{4} - \dfrac{x^4}{16} + C$]

Int. by Parts

$u = \ln x \qquad v = \frac{x^4}{4}$

$u' = 1/x \qquad v' = x^3$

$$\int x^3 \ln x \, dx = (\ln x)\left(\frac{x^4}{4}\right) - \int \left(\frac{1}{x}\right)\left(\frac{x^4}{4}\right) dx$$

$$= \frac{x^4}{4} \ln x - \int \frac{x^3}{4} dx$$

$$= \frac{x^4}{4} \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + C$$

$$= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$

4. If an egg is thrown upward at 20 feet per second from a height of 10 feet [and acceleration due to gravity is 32 feet per second^2 downward],

(a) Find formulas for the egg's velocity and height after t seconds

(b) How fast is the egg moving when it hits the ground?

$h(t) = -16t^2 + 20t + h_0$ ← h_0 should be 10 because of starting height

$v(t) = -32t + v_0$ ← v_0 should be 20 since it started out upward 20 ft/sec

$a(t) = -32$

antiderivative of $a(t)$

antiderivative of $v(t)$

$$h(t) = -16t^2 + 20t + 10$$
$$v(t) = -32t + 20$$

5. Evaluate $\int \frac{1}{5-3x} dx$.

let $u = 5-3x$

$\frac{du}{dx} = -3$

$\frac{du}{-3} = dx$

$= \int \frac{1}{u} \cdot \frac{du}{-3}$

$= \frac{-1}{3} \ln|u| + C$

$= \frac{-1}{3} \ln|5-3x| + C$

6. If $F(x) = \int_x^{\pi} \sqrt{\sin t}\, dt$, what is $F'(x)$?

$F(x) = -\int_{\pi}^{x} \sqrt{\sin t}\, dt$

$F'(x) = -\sqrt{\sin x}$

By Fun. Thrm. of Calc. pt. 1

7. Integrate $\int \sin^n x \cos^3 x\,dx$. [Hint: the answer is $\frac{\sin^{n+1} x}{n+1} - \frac{\sin^{n+3}}{n+3} + C$]

$$= \int \sin^n x \cdot \cos^2 x \cdot \cos x\,dx$$

$$= \int \sin^n x \cdot (1 - \sin^2 x) \cdot \cos x\,dx$$

$$= \int (\sin^n x - \sin^{n+2} x) \cos x\,dx$$

let $u = \sin x$

$\frac{du}{dx} = \cos x$

$\frac{du}{\cos x} = dx$

$$= \int (u^n - u^{n+2}) \cdot \cos x \cdot \frac{du}{\cos x}$$

$$= \int (u^n - u^{n+2})\,du$$

$$= \frac{u^{n+1}}{n+1} - \frac{u^{n+3}}{n+3} + C$$

$$= \frac{\sin^{n+1} x}{n+1} - \frac{\sin^{n+3} x}{n+3} + C$$

8. Because of a stunningly negligent editor named Brian, the first printing of the second edition of CliffsQuickReview Calculus was released with a table of integrals that said $\int \frac{dx}{a^2+x^2} = \frac{1}{2}\arctan\frac{x}{2} + C$. **Explain**, in simple enough terms that Brian can follow along (Brian claimed to have taken calculus himself), exactly how this formula is or is not acceptable and why.

First of all, it's suspicious that the left side has "a" but the right side doesn't.

But the real way to make sure is check it. If the antiderivative of $\frac{1}{a^2+x^2}$ is $\frac{1}{2}\arctan\frac{x}{2}+C$, then the derivative of $\frac{1}{2}\arctan\frac{x}{2}+C$ would be $\frac{1}{a^2+x^2}$.

But if we take the derivative, we have

$$\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C\right]' = \frac{1}{2}\cdot\frac{1}{1+\left(\frac{x}{2}\right)^2}\cdot\frac{1}{2}$$

$$= \frac{1}{4}\cdot\frac{1}{1+\frac{x^2}{4}}$$

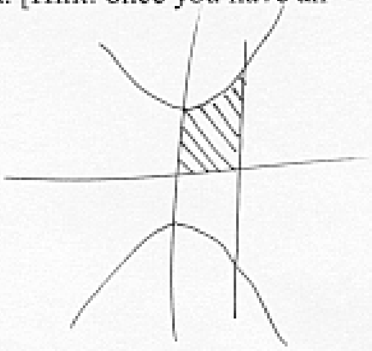
$$= \frac{1}{4}\cdot\frac{1}{\frac{4+x^2}{4}}$$

$$= \frac{1}{4}\cdot\frac{4}{4+x^2}$$

$$= \frac{1}{4+x^2}$$

which isn't quite $\frac{1}{a^2+x^2}$, so it looks like the moron turned the "a" into a "2".

9. The graph of $y^2 - x^2 = 4$ is a hyperbola. Set up an integral for the area in the first quadrant bounded by the hyperbola and the line $x = 1$ and use it to find the area. [Hint: once you have an integral set up the substitution $x = 2\tan\theta$ might be useful].



Solve for y: $y^2 = 4 + x^2$

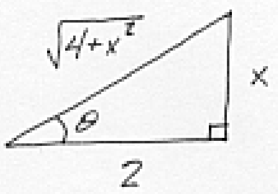
$$y = \pm\sqrt{4+x^2}$$

$$\text{Area} = \int_0^1 \sqrt{4+x^2}\,dx$$

let $x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2}$

$$\frac{dx}{d\theta} = 2\cdot\sec^2\theta$$

$$dx = 2\sec^2\theta\,d\theta$$



so $\sin\theta = \frac{x}{\sqrt{4+x^2}}$

$\cos\theta = \frac{2}{\sqrt{4+x^2}}$

$$= \int_{x=0}^{x=1} \sqrt{4+(2\tan\theta)^2}\cdot 2\sec^2\theta\,d\theta$$

$$= \int_{x=0}^{x=1} \sqrt{4+4\tan^2\theta}\cdot 2\sec^2\theta\,d\theta$$

$$= \int_{x=0}^{x=1} 2\sqrt{1+\tan^2\theta}\cdot 2\sec^2\theta\,d\theta$$

$$= \int_{x=0}^{x=1} 4\cdot\sqrt{\sec^2\theta}\cdot\sec^2\theta\,d\theta$$

$$= 4\int_{x=0}^{x=1} \sec^3\theta\,d\theta$$

By Table line 21

$$= 4\left[\frac{1}{2}\,\frac{\sin\theta}{\cos^2\theta} + \frac{1}{2}\int \sec\theta\,d\theta\right]_{x=0}^{x=1}$$

By Table line 22

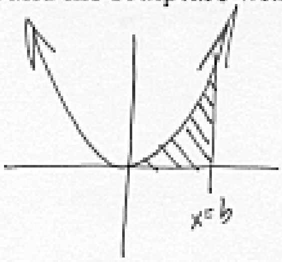
$$= \left[2\,\frac{\sin\theta}{\cos^2\theta} + 2\cdot\frac{1}{2}\ln\left|\frac{\sin\theta+1}{\sin\theta-1}\right|\right]_{x=0}^{x=1}$$

$$= \left[2\cdot\frac{\frac{x}{\sqrt{4+x^2}}}{\left(\frac{2}{\sqrt{4+x^2}}\right)^2} + \ln\left|\frac{\frac{x}{\sqrt{4+x^2}}+1}{\frac{x}{\sqrt{4+x^2}}-1}\right|\right]_0^1$$

$$= \left(\frac{2\cdot\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} + \ln\left|\frac{\frac{1}{\sqrt{5}}+1}{\frac{1}{\sqrt{5}}-1}\right|\right) - \left(\frac{2\cdot 0}{1} + \ln\left|\frac{0+1}{0-1}\right|\right)$$

Don't be freaked! It's a hard problem. I won't put something this long on the actual test.

10. Jon wants to have a huge mathematical sculpture built in the center of the quad. It's going to be shaped like the region under $y = x^2$, and of course above $y = 0$, between $x = 0$ and $x = b$, but Jon can't decide how big b should be. If he can only afford 10 square meters of steel plate to build his sculpture with, how big should he make it?



$$\text{Area} = \int_0^b x^2\,dx$$

$$= \left.\frac{x^3}{3}\right|_0^b$$

$$= \frac{b^3}{3} - \frac{0^3}{3}$$

$$= \frac{b^3}{3}$$

So for the area to be 10, we'd have

$$10 = \frac{b^3}{3}$$

$$30 = b^3$$

$$b = \sqrt[3]{30}$$

Extra Credit (5 points possible):
If a dart is thrown at a square dart board so that it hits at a random spot, what is the probability that it hits closer to the center than the edge? [It's really a question about areas, if you think about it right.]