**Practice Exam 2 Calc 2 2/17/2004**

Each problem is worth 10 points. For full credit provide complete justification for your answers. Warning: This is only a practice exam. Knowing how to do the problems on this exam is good, but if that's **all** you know, you're not likely to do well at all on the actual exam. Use this as a warm-up, a trial run, or just to get used to the form questions might take, but don't think of it as a complete study guide.

1. Suppose that cars pass Coe's campus on First Avenue with a density of 200 cars per hour at 4am one morning, and this increases linearly to 1000 cars per hour at 6am. How many cars pass during the period from 4am to 6am?

$$m = \frac{1000-200}{2-0} = 400$$

$$\delta(t) = 200 + 400t$$

(in hours after 4am)

$$\text{Total Cars} = \int_0^2 (200+400t)\,dt$$

$$= \left[200t + 200t^2\right]_0^2$$

$$= 400 + 800 - 0$$

$$= 1200 \text{ cars}$$

2. If the work required to stretch a spring 1 foot beyond its natural length is 15 ft-lb, how much work is needed to stretch it 6 inches beyond its natural length?

We know: $\int_0^1 kx\,dx = 15$

Hooke's Law

$$\left.\frac{kx^2}{2}\right|_0^1 = 15$$

$$\frac{k}{2} = 15$$

$$k = 30$$

So $f = 30x$ for this spring...

So we can find: $\int_0^{1/2} 30x\,dx = \left.15x^2\right|_0^{1/2}$

$$= \frac{15}{4} - 0$$

$$= 3.75 \text{ ft-lb}$$

3. You have a big opportunity to buy an ostrich ranch. If you expect a revenue of \$50,000 per year to start immediately and increase steadily up to a level of \$100,000 per year after 10 years, write an integral which gives the present value of the revenue from the ranch over the next 10 years (assume 6% continuously compounded interest).

$$m = \frac{100000 - 50000}{10 - 0} = 5000$$

So $P(t) = 50000 + 5000t$

4. Set up an integral for the length of the curve $y = \sin x$ between the point $(0,0)$ and the point $(\pi,0)$.

$$\text{Arc Length} = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \qquad y' = \cos x$$

$$= \int_0^{\pi} \sqrt{1 + (\cos x)^2}\, dx$$

$$= \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx$$

5. If the probability density function for the length of time it takes a baby lynx to stray more than 1 foot from its mother after birth is given by $p(t) = 0.32e^{-0.32t}$, where $t$ is measured in days, what is the median time it takes a baby lynx to stray more than 1 foot from its mother?

The median is the value of c for which

$$\int_0^c .32e^{-.32t}\,dt = \frac{1}{2}$$

let $u = -.32t$

$\frac{du}{dt} = -.32$

$\frac{du}{-.32} = dt$

$$\int_{t=0}^{t=c} .32e^u \cdot \frac{du}{-.32} = \frac{1}{2}$$

$$-e^u \Big|_{t=0}^{t=c} = \frac{1}{2}$$

$$-e^{-.32t} \Big|_0^c = \frac{1}{2}$$

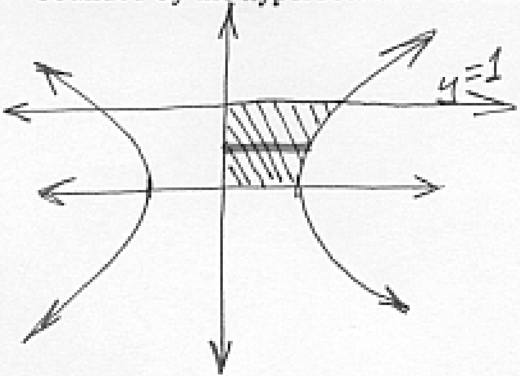
$$-e^{-.32c} + e^0 = \frac{1}{2}$$

$$e^{-.32c} = \frac{1}{2}$$

$$-.32c = \ln \frac{1}{2}$$

$c \approx 2.17$ days

6. The graph of $x^2 - y^2 = 1$ is a hyperbola. Set up an integral for the area in the first quadrant bounded by the hyperbola and the line $y = 1$ and use it to find the area.




Width of a slice $= \sqrt{1+y^2} - 0$

Area of a slice $= \sqrt{1+y^2}\cdot\Delta y$

Total Area $= \int_0^1 \sqrt{1+y^2}\,dy$

Table line 30

$$= \frac{1}{2}\left(y\sqrt{1+y^2} + \int \frac{1}{\sqrt{1+y^2}}\,dy\right)\Bigg|_0^1$$

Table line 29

$$= \frac{1}{2}\left[y\sqrt{1+y^2} + \ln\left|y+\sqrt{1+y^2}\right|\right]_0^1$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln(1+\sqrt{2})\right] - \frac{1}{2}\left[0 + \ln 1\right]$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln(1+\sqrt{2})$$

$x^2 - y^2 = 1$

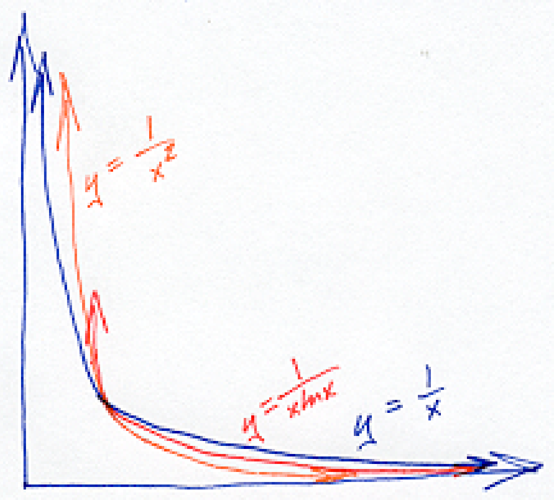
$x^2 = 1 + y^2$

$x = \pm\sqrt{1+y^2}$

7. Brandi is a calculus student at E.S.U. who's having some trouble with improper integrals. Brandi says "So there was this problem on our test, and it was to say if one over $x \ln x$ converged or not. I drew this total blank on how to find the antiderivative, but I thought about it and decided I didn't really have to. See, you know that if you integrate $1/x$ it's infinity, but if you integrate $1/x^2$ or anything else where the denominator is more than just $x$ it converges. So I just said all that and said it must converge. But the grader gave me no points at all, and just said that didn't work, but not why, and then he told me to go away because he hates dealing with students."

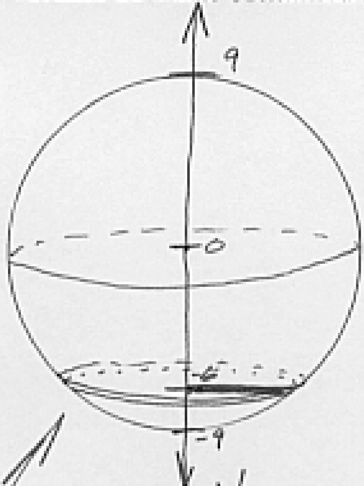
Explain to Brandi either why she's right, or what's wrong with her reasoning.

Well Brandi, first of all, to find $\int_2^\infty \frac{1}{x\ln x}\,dx$ you should just antidifferentiate by letting $u = \ln x$ and go from there. But that doesn't really get at what's wrong with what you tried. It sounds like you were thinking of the rule from class that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges if $p$ is 1 or less, but converges if $p$ is greater than 1. You thought that having the $\ln x$ down there made it like an exponent bigger than 1, so it would converge. That doesn't work.




If you look at this graph, what we know is that the area under the blue graph, $y = 1/x$, is infinite as you go out to the right. We also know that the area under the orange graph, $y = 1/x^2$, is finite as you go out to the right. But really all your comparison does is show that the red graph, $y = \frac{1}{x\ln x}$, lies between the other two and thus has an area between the other two – and being less than or equal to infinity doesn't tell us much at all!

8. Jon has a bowl-shaped fountain on his desk which is shaped like a frustum of a sphere with a radius of 9 inches, cut off 3 inches up from the bottom. Set up an integral and use it to find the volume of water contained in this fountain.




Slices are horizontal, shaped like circular cylinders

$$x^2 + y^2 = 9^2$$

$$x = \pm\sqrt{9^2 - y^2}$$

Radius of a slice $= \sqrt{9^2 - y^2}$ in.

Area of a slice $= \pi\left(\sqrt{9^2 - y^2}\right)^2$ in.$^2$

Vol. of a slice $= \pi\left(9^2 - y^2\right)\Delta y$ in.$^3$

$$\text{Total Volume} = \int_{-9}^{-6} \pi\left(9^2 - y^2\right)dy$$

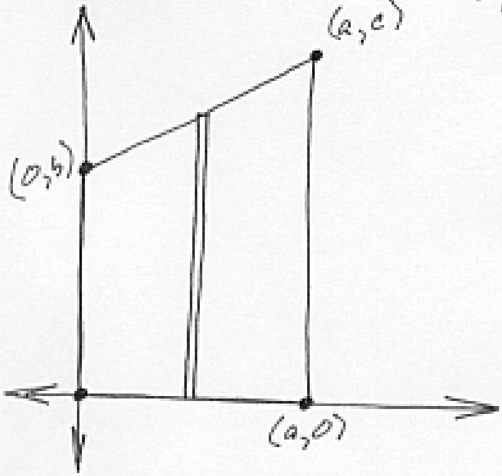
$$= \pi\left[9^2 y - \frac{1}{3}y^3\right]_{-9}^{-6}$$

$$= \pi\left[(-486 + 72) - (-729 + 243)\right]$$

$$= \pi(-414 + 486)$$

$$= 72\pi \text{ in}^3$$

9. **Find** the $x$ coordinate of the centroid of the trapezoidal region with vertices at (0,0), ($a$,0), (0,$b$), and ($a$,$c$).




Equation of top boundary: $m = \dfrac{c-b}{a-0}$ intercept $b$

$$y = \frac{c-b}{a}x + b$$

So height of a slice $= \dfrac{c-b}{a}x + b$

area of a slice $= \left(\dfrac{c-b}{a}x + b\right)\Delta x$

$$\text{Total area} = \int_0^a \left(\frac{c-b}{a}x + b\right)dx$$

$$= \left[\frac{c-b}{a}\cdot\frac{x^2}{2} + bx\right]_0^a$$

$$= \left(\frac{c-b}{a}\cdot\frac{a^2}{2} + b\cdot a\right) - (0+0)$$

$$= \frac{1}{2}ac - \frac{1}{2}ab + ab$$

$$= \frac{1}{2}ac + \frac{1}{2}ab$$

$$= \frac{a(b+c)}{2}$$

$$\text{Moment} = \int_0^a x\left(\frac{c-b}{a}x + b\right)dx$$

$$= \int_0^a \left(\frac{c-b}{a}x^2 + bx\right)dx$$

$$= \left[\frac{c-b}{a}\,\frac{x^3}{3} + b\,\frac{x^2}{2}\right]_0^a$$

$$= \left(\frac{c-b}{a}\cdot\frac{a^3}{3} + b\cdot\frac{a^2}{2}\right) - (0+0)$$

$$= \frac{1}{3}a^2c - \frac{1}{3}a^2b + \frac{1}{2}a^2b$$

$$= \frac{1}{3}a^2c + \frac{1}{6}a^2b$$

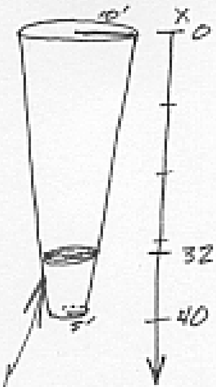
$$= \frac{a^2}{6}(2c+b)$$

$$\bar{x} = \frac{\text{Moment}}{\text{Area}}$$

$$= \frac{\frac{a^2}{6}(2c+b)}{\frac{a(b+c)}{2}}$$

$$\bar{x} = \frac{a(b+2c)}{3(b+c)}$$

10. A well with circular horizontal cross-sections is 40 feet deep and narrows from 10 feet in diameter at the surface to 6 feet in diameter at its bottom. If the well has 8 feet of water at the bottom, how much water will 1000 foot-pounds of work pump to the top of the well?



Slices are horizontal, shaped like circular cylinders

| $x$ | $r$ |
|---|---|
| 0 | 10 |
| 40 | 5 |

$m = \frac{5-10}{40-0} = \frac{-5}{40} = \frac{-1}{8}$

$r = \frac{-1}{8}x + 10$

Radius of a slice $= \left(10 - \frac{x}{8}\right)$ ft

Area of a slice $= \pi\left(10 - \frac{x}{8}\right)^2$ ft$^2$

Vol. of a slice $= \pi\left(10 - \frac{x}{8}\right)^2 \Delta x$ ft$^3$

Weight of a slice $= \pi\left(\frac{x^2}{64} - \frac{5x}{2} + 100\right)\Delta x$ ft$^3 \cdot 62.4$ lb/ft$^3$

Work for a slice $= 62.4\pi\left(\frac{x^2}{64} - \frac{5x}{2} + 100\right)\Delta x$ lb $\cdot\, x$ ft

So we need to solve: $\int_{32}^{c} 62.4\pi\left(\frac{x^3}{64} - \frac{5x^2}{2} + 100x\right)dx = 1000$

$$62.4\pi\left[\frac{x^4}{256} + \frac{5x^3}{6} + 50x^2\right]_{32}^{c} = 1000$$

$$\left(\frac{c^4}{256} + \frac{5c^3}{6} + 50c^2\right) - \left(4096 + \frac{81920}{3} + 51200\right) = \frac{1000}{62.4\pi}$$

$$\frac{c^4}{256} + \frac{5c^3}{6} + 50c^2 = 82607.8$$

And my calculator says $c \approx 32.678$ is the solution to this, so about .678 feet, or just over 8 inches, will be pumped out.