Exam 3b Calc 2 3/26/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find the sum of the series $1-\underbrace{\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\ldots}_{\text{Geo series}}$

$\underline{a} = 1$

$r = \dfrac{-2/3}{1} = -\dfrac{2}{3}$

Sum of the series for infinite terms $= \dfrac{a}{1-r}$ for $|r|<1$

$= \dfrac{1}{1-(-2/3)}$

Good

$= \dfrac{1}{1+2/3}$

$= \dfrac{3}{3+2} = \dfrac{3}{5}$ // Ans

2. Write the series $(x-2)+\dfrac{(x-2)^2}{3}+\dfrac{(x-2)^3}{5}+\dfrac{(x-2)^4}{7}+\ldots$ in sigma notation.

$$\boxed{\sum_{n=1}^{\infty} \frac{(x-2)^n}{n+(n-1)}}$$ Works.

For $n=1$: $\dfrac{(x-2)^1}{1+(1-1)} = \dfrac{(x-2)}{1}$ ✓

For $n=2$: $\dfrac{(x-2)^2}{2+(2-1)} = \dfrac{(x-2)^2}{3}$ ✓

For $n=3$: $\dfrac{(x-2)^3}{3+(3-1)} = \dfrac{(x-2)^3}{5}$ ✓

For $n=4$: $\dfrac{(x-2)^4}{4+(4-1)} = \dfrac{(x-2)^4}{7}$ ✓

3. Write a fifth degree Taylor polynomial for $\frac{\cos x - 1}{x^2}$ centered at zero.

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \frac{x^8}{8!}$$

$$\cos x - 1 = \cancel{1} - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \cancel{-1}$$

$$\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\frac{\cos x - 1}{x^2} = -\frac{\cancel{x^2}}{2 \cdot \cancel{x^2}} + \frac{x^{\cancel{4}2}}{4!\,\cancel{x^2}} - \frac{x^{\cancel{6}4}}{6!\,\cancel{x^2}} + \dots$$

$$\frac{\cos x - 1}{x^2} \approx -\frac{1}{2} + \frac{x^2}{24} - \frac{x^4}{720}$$

Great

4. The first three derivatives of the function $f(x) = \arccos x$ are listed below. Use them to find the 3rd degree Taylor polynomial for $\arccos x$ centered at $x = 0$.

$$f'(x) = \frac{-1}{\sqrt{1-x^2}}$$

$$f''(x) = \frac{-x}{\left(1-x^2\right)^{3/2}}$$

$$f'''(x) = \frac{-1}{\left(1-x^2\right)^{3/2}} - \frac{3x^2}{\left(1-x^2\right)^{5/2}}$$

$f(x) = \arccos x$

$f(0) = \pi/2$

$f'(0) = -1$

$f''(0) = 0$

$f'''(0) = -1$

$$P_3(x) = \pi/2 - 1x + \cancel{0x^2/2} - x^3/3!$$

$$P_3(x) = \pi/2 - x - x^3/3!$$

Great

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ converges or diverges.

comparison test!!

compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$n^2+2 > n^2$, so $\frac{1}{n^2+2} < \frac{1}{n^2}$

we know that for any value of $p > 1$ in $\sum_{n=1}^{\infty} \frac{1}{n^p}$ the series converges $2 > 1$, so $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Any series less than a converging series also converges, so $\sum_{n=1}^{\infty} \frac{1}{n^2+2}$ converges

Well done!

6. Is x = 1 included in the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$?

$\frac{1^n}{n^2}$ always $= \frac{1}{n^2}$ for any value of n. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p series, so $x = 1$ is included in the interval

Excellent

7. What is the radius of convergence of the series $\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$?

$\sum_{n=0}^{\infty} \left(\frac{x}{4}\right)^n$

Ratio test

$\lim_{n \to \infty} \left| \frac{x^{n+1}}{4^{n+1}} \cdot \frac{4^n}{x^n} \right|$

$= \lim_{n \to \infty} |x| \cdot \frac{1}{4}$

Hence, the series will converge only if $\frac{1}{4}|x| < 1$

$|x| < 4$ or, $-4 < x < 4$

So, radius of convergence is 4.

Great Job

8. Biff is having trouble with calculus again. He says "Dude, these series are kicking my ass. There was this one thing, where, like, we were supposed to say something about using a series to approximate something, right? And like, what would happen if you were putting in something outside the interval of convergence, right? So I said that then you'd have to use really a lot of terms, like a really high degree, to make it accurate, you know? But the guy who sits next to me said he didn't think that was right."

Explain to Biff whether using a higher degree polynomial approximation will help outside the radius of convergence, and why.

Biff, I think that using a higher degree polynomial approximation won't help outside the radius of convergence Because, a taylor polynomial gets close to the function only at the radius of convergence. And if you go beyond the radius of convergence your polynomial would go towards different direction. So even if you take a high degree polynomial or a low degree polynomial both of these help approximating only inside the radius of convergence.

Well put!

9. Does the series $1-\frac{2}{3}+\frac{3}{5}-\frac{4}{7}+\frac{5}{9}-\ldots$ converge? Justify your answer.

$$\sum_{n=1} (-1)^{n+1}\frac{n}{2n-1}$$

(0) Alternates ✓

(1) Is $\frac{n+1}{2(n+1)-1} < \frac{n}{2n-1}$?

Look at $f(x) = \frac{x}{2x-1}$

$$f'(x) = \frac{1(2x-1) - x\cdot 2}{(2x-1)^2} = \frac{-1}{(2x-1)^2}$$

The numerator is negative, and the denominator is a square so it's positive, which makes the fraction negative. Thus the derivative is always negative.

Decreasing? ✓

(2) Is $\lim_{n\to\infty} \frac{n}{2n-1} = 0$?

It's $\frac{\infty}{\infty}$ indeterminate, so

$$\stackrel{L'H}{=} \lim_{n\to\infty} \frac{1}{2}$$

Wait! The limit isn't zero! So we keep adding and then subtracting things close to ½, and by the test for divergence this series diverges.

10. Find the fifth degree Taylor polynomial for sin(tan x). [Hint: The derivatives involved are ridiculous – find a better way.]

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots \qquad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

$$\text{So } \sin(\tan x) \approx \left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right) - \frac{\left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right)^3}{6} + \frac{\left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right)^5}{120}$$

$$= x + \frac{x^3}{3} + \frac{2x^5}{15} - \frac{x^3}{6} - \frac{x^5}{3} \cdot \frac{1}{6} \cdot 3 + \frac{x^5}{120} + \text{other crap of high degree}$$

$$= x + \frac{x^3}{6} - \frac{x^5}{40}$$