Exam 4a Calc 2 4/23/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Determine whether $y = 3e^{-4t}$ is a solution to the differential equation $\frac{d^2y}{dt^2} = -4y$.

$y' = -12e^{-4t}$

$y'' = 48e^{-4t}$

$y'' = -4y$

$48e^{-4t} = -4(3e^{-4t})$

$48e^{-4t} \neq -12e^{-4t}$

Good

No this is not a solution

2. Write a differential equation representing a population with logistic growth and a carrying capacity of 500 (use k for the constant of proportionality).

$\frac{dL}{dt} = KP(P-500)$

Yep

3. Find a general solution to the differential equation $\frac{dy}{dx} + x^2 y^2 = 0$.

$$\frac{dy}{dx} = -x^2 y^2$$

01, $\int \frac{1}{y^2}\,dy = \int -x^2\,dx$

01, $\frac{y^{-2+1}}{-2+1} = -\frac{x^{2+1}}{2+1} + C$

01, $\frac{y^{-1}}{-1} = -\frac{x^3}{3} + C$

01, $-\frac{1}{y} = -\frac{x^3}{3} + C$

01, $-\frac{1}{y} = \frac{-x^3+3C}{3}$

01, $-y = \frac{3}{-x^3+3C}$

01, $y = -\left(\frac{3}{-x^3+3C}\right) = \frac{-3}{3C-x^3}$

$\boxed{y = \frac{-3}{3C-x^3}}$ is the general solution, where C is a constant.

Great Job

4. Find a general solution for the differential equation $y'' - 7y' + 6y = 0$.

$$y'' - 7y' + 6y = 0$$

So $y = e^{st}$

$y' = se^{st}$

$y'' = s^2 e^{st}$

$$s^2 e^{st} - 7se^{st} + 6e^{st} = 0$$

Factor out e^{st}

$$e^{st}(s^2 - 7s + 6) = 0 \qquad e^{st} \neq 0$$

so

$s^2 - 7s + 6 = 0$ factor

$$(s-1)(s-6) = 0$$

$s = 1$ + $s = 6$

so $y = e^{t}$ Yes and $y = e^{6t}$

Excellent

so we have

$$\boxed{y = Ae^{t} + Be^{6t}}$$

5. Find all equilibrium points of the system

$$\frac{dx}{dt} = x - x^2 - \frac{xy}{3}$$

$$\frac{dy}{dt} = y - y^2 - \frac{xy}{2}$$

$0 = x\left(1 - x - \frac{y}{3}\right) \Rightarrow x = 0$ or $1 - x - \frac{y}{3} = 0$

$0 = y\left(1 - y - \frac{x}{2}\right)$

$\Downarrow$

$y = 0$ or $1 - y = 0$

$\Downarrow$

$y = 1$

Equilibrium points:

$(0,0), (0,1), (1,0), (4/5, 3/5)$

$x = 1 - y/3$

$\Downarrow$

$y = 0$ or $1 - y - \frac{1 - y/3}{2} = 0$

$1 - y - \frac{1}{2} + \frac{y}{6} = 0$

$\frac{1}{2} - \frac{5}{6}y = 0$

$\frac{6}{10} = y$

6. Water leaks from a vertical cylindrical tank through a small hole in its base at a rate proportional to the square root of the volume of water remaining. If the tank initially contains 100 liters and 20 liters leak out during the first day, when will the tank be half empty? correct to 2 decimal places

80 left

$$\frac{dw}{dt} = k\sqrt{w}$$

$$\int \frac{1}{\sqrt{w}}\,dw = \int k\,dt$$

$$\int w^{-1/2}\,dw = \int k\,dt$$

$$2w^{1/2} = kt + C \qquad A = \tfrac{1}{2}C$$

$$w^{1/2} = \tfrac{1}{2}kt + \tfrac{1}{2}C$$

$$w = \left(\tfrac{1}{2}kt + A\right)^2$$

t is in days

$w(0) = 100$

$w(1) = 80$

$$100 = \left(\tfrac{1}{2}k(0) + A\right)^2$$

$$100 = A^2 \qquad A = 10$$

$$80 = \left(\tfrac{1}{2}k(1) + 10\right)^2$$

$$\sqrt{80} = \tfrac{1}{2}k + 10$$

$$\sqrt{80} - 10 = \tfrac{1}{2}k$$

$$2\sqrt{80} - 20 = k = -2.11$$

$$w(t) = \left(\tfrac{1}{2}(-2.11)t + 10\right)^2$$

$\frac{1}{2}$ empty means 50 liters left

$$50 = \left(\tfrac{1}{2}(-2.11)t + 10\right)^2$$

$$\sqrt{50} = -1.056t + 10$$

$$\sqrt{50} - 10 = -1.056t$$

$$t = \frac{\sqrt{50} - 10}{-1.056}$$

$$\approx \boxed{2.77 \text{ days}}$$

Excellent

8. Lake Erie has a volume of 460,000 km^3 and an outflow rate of 175 km^3 per year. Suppose that 20kg of a certain pollutant is present in the lake at time 0, when a massive filtration project is undertaken to remove the substance from the lake. If 0.1kg per year is removed from the lake beyond the amount removed by natural processes,

a) Write a differential equation for the amount of pollutant in the lake after t years.

b) Use Euler's method with $\Delta t = 5$ to estimate the pollutant remaining in the lake after 10 years.

$$\frac{dQ}{dt} = \frac{-r}{V} \cdot Q - 0.1$$

$$\frac{dQ}{dt} = \frac{-175}{460000} Q - 0.1$$

$$Q(0) = 20$$

$$\frac{dQ}{dt} = -.0003804 \cdot Q - 0.1$$

$$\Delta Q = -.107609 \cdot \Delta t$$

$$\Delta Q = -.53804$$

So $Q(5) = 19.4619565$

$$\frac{dQ}{dt} = -.0003804 \cdot Q - 0.1$$

$$\Delta Q = -1.07404 \cdot \Delta t$$

$$\Delta Q = -.53702$$

So $Q(10) = 18.92$ $\quad$ (18.9249364957)

9. Show that the function $y = te^{-t}$ is a solution to any differential equation of the form

$$ay'' + 2ay' + ay = 0$$

regardless of the value of the constant a.

$y = te^{-t}$

$y' = e^{-t} - te^{-t}$

$y'' = -e^{-t} - (e^{-t} - te^{-t})$

$y'' = te^{-t} - 2e^{-t}$

$a(te^{-t} - 2e^{-t}) + 2a(e^{-t} - te^{-t}) + a(te^{-t}) = 0$

$ae^{-t}(t - \cancel{2} + \cancel{2} - 2\cancel{t} + t) = 0$

$ae^{-t}(0) = 0$

$0 = 0$ ✓

Excellent!

10. Find a general solution to the differential equation $y'' + \omega^2 y = 0$, where ω is a constant.

suppose $y = e^{st}$ is a sol.

$y' = se^{st}$

$y'' = s^2 e^{st}$

$s^2 e^{st} + \omega^2 e^{st} = 0$

$e^{st}(s^2 + (0s) + \omega^2) = 0$

W

$\dfrac{-\cancel{0} \pm \sqrt{0^2 - 4 \cdot 1 \cdot \omega^2}}{2}$

$\dfrac{\pm\sqrt{-4}\,\omega^2}{2} = \dfrac{\pm\omega\sqrt{-1 \cdot 4}}{2} = \pm\dfrac{2\omega\sqrt{-1}}{2}$ $\pm \omega i$

$e^{ix} = \cos x + i\sin x$

$y = e^{\omega i t}$

$y = e^{-\omega i t}$

$y = Ae^{\omega i t} + Be^{-\omega i t}$

$e^{\omega i t} = \cos \omega t + i \sin \omega t$

$e^{-\omega i t} = \cos -\omega t + i \sin -\omega t$

same as

$\cos \omega t - i \sin \omega t$

Excellent

$y = A[\cos \omega t + i \sin \omega t] + B[\cos \omega t - i \sin \omega t]$

$y = A\cos \omega t + B\cos \omega t + Ai\sin \omega t - Bi\sin \omega t$

$y = \underbrace{(A+B)}_{C}\cos(\omega t) + \underbrace{(Ai - Bi)}_{D}\sin(\omega t)$

$\boxed{y = C\cos(\omega t) + D\sin(\omega t)}$