Practice Exam 4 Calc 2 3/26/2004

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find all equilibrium population values for the differential equation $\frac{dp}{dt} = 0.03p(45-p)$.

Equilibrium means no change, so $\frac{dp}{dt} = 0$, or:

$$0 = 0.03p(45-p)$$

But that product is only zero if $p=0$ or $45-p=0$,
So we have equilibrium when $p=0$ or $p=45$

2. a) Find a general solution to the differential equation $\frac{dy}{dx} + xy^2 = 0$.

b) Find a particular solution to the differential equation from part a satisfying the condition $y(1) = 1$.

I. separate

$$\frac{dy}{dx} = -xy^2$$

II. integrate

$$\int \frac{1}{y^2}\,dy = -\int x\,dx$$

III. solve

$$-y^{-1} = -\frac{x^2}{2} + C$$

$$\frac{1}{y} = \frac{x^2}{2} + C$$

$$\frac{1}{y} = \frac{x^2+2C}{2}$$

a) $y = \frac{2}{x^2+2C}$

Since $y(1)=1$:

$$(1) = \frac{2}{(1)^2+2\cdot C}$$

$$1+2C = 2$$

$$2C = 1$$

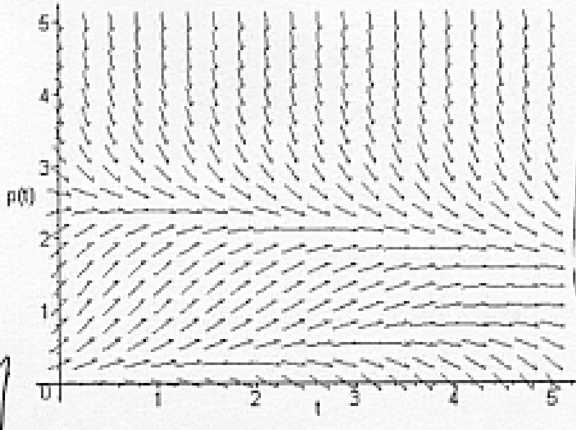
$$C = 1/2$$

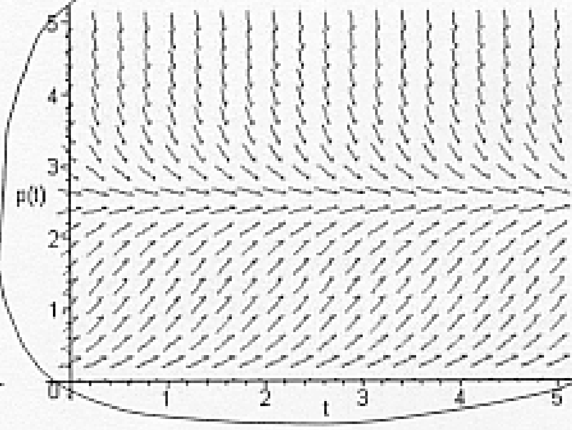
So $y = \frac{2}{x^2+2\cdot 1/2}$

b) or $y = \frac{2}{x^2+1}$

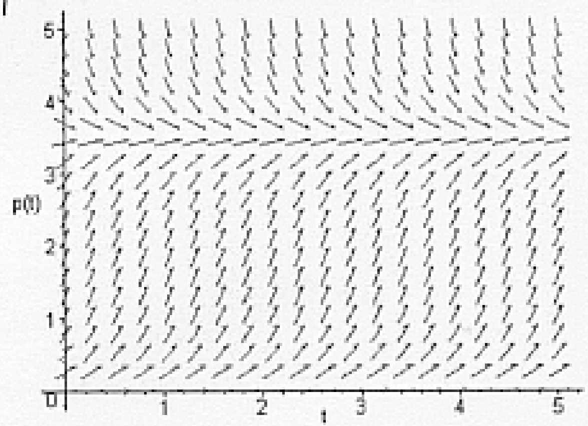
3. Which of the slope fields pictured below could represent the differential equation

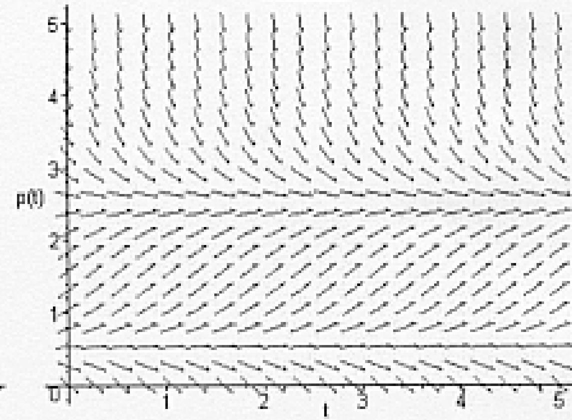
$$\frac{dP}{dt} = -P(P-2.5)?$$





This one





This one has no equilibria

This one's top equilibrium is too high

This one has an equilibrium at $p \approx .500$

This one looks just right, logistic growth with equilibrium at $p = 0$ or at $p = 2.5$.

4. For the differential equation $y'' - 5y' + 6y = 0$,

a) Find a general solution

b) Find the particular solution satisfying the conditions $y(0) = 0$ and $y'(0) = 2$.

Let's look for solutions of the form $y = e^{s \cdot t}$, so

$$y' = s \cdot e^{st}$$
$$y'' = s^2 \cdot e^{st}$$

Then we need: $(s^2 \cdot e^{st}) - 5(s \cdot e^{st}) + 6(e^{st}) = 0$

or: $e^{st}(s^2 - 5s + 6) = 0$

$$e^{st}(s-3)(s-2) = 0$$

So $s = 3$ or $s = 2$

Then the general solution is:

a) $y = a \cdot e^{3t} + be^{2t}$

Since $y(0) = 0$:

$(0) = a \cdot e^{3(0)} + b \cdot e^{2(0)}$

or $0 = a + b$

or $a = -b$

$a = -(-2)$

$a = 2$

Since $y'(0) = 2$:

$y' = 3ae^{3t} + 2be^{2t}$

$(2) = 3ae^{3(0)} + 2be^{2(0)}$

$2 = 3a + 2b$

$2 = 3(-b) + 2b$

$2 = -b$

$b = -2$

b) So $y = 2e^{3t} - 2e^{2t}$ is our particular solution

5. For the differential equation $y'' + 5y = 0$,

a) Find a general solution

b) Find the particular solution satisfying the conditions $y(0) = 0$ and $y'(0) = 2$.

Let's try $y = e^{s \cdot t}$

$y' = s \cdot e^{st}$

$y'' = s^2 \cdot e^{st}$

So we need $(s^2 \cdot e^{st}) + 5(e^{st}) = 0$

$e^{st}(s^2 + 5) = 0$

Use quadratic formula:

$$s = \frac{-(0) \pm \sqrt{(0)^2 - 4(1)(5)}}{2(1)}$$

$$= \frac{\pm 2i\sqrt{5}}{2}$$

$$= \pm i\sqrt{5}$$

So the general solution can be written $y = a \cdot e^{i\sqrt{5}t} + be^{-i\sqrt{5}t}$

a) But that's just a silly way to write $y = k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t)$

Since $y(0) = 0$:

$0 = k_1 \cos(0) + k_2 \sin(0)$

$0 = k_1 \cdot 1$

$k_1 = 0$

Since $y'(0) = 2$:

$y' = -k_1 \cdot \sqrt{5} \cdot \sin(\sqrt{5}t) + k_2\sqrt{5}\cos(\sqrt{5}t)$

$2 = -k_1\sqrt{5} \cdot 0 + k_2 \cdot \sqrt{5} \cdot 1$

$k_2 = \frac{2}{\sqrt{5}}$

And that means our particular solution is:

$$y = 0\cos(\sqrt{5}t) + \frac{2}{\sqrt{5}}\sin(\sqrt{5}t)$$

6. If the differential equation $\frac{dT}{dt} = 0.05(400 - T)$ represents the temperature after t minutes of a toy mouse (which starts out at 70°) that Jon's cat Nemo drops into the oven, **use Euler's method** with $\Delta t = 5$ to estimate the temperature of the mouse ten minutes later when Jon notices that something smells funny.

We start with $T(0) = 70°$. At this point

$$\frac{dT}{dt} = 0.05(400° - 70°) \approx 16.5°$$

So $\Delta T = 16.5 \cdot \Delta t = 16.5 \cdot (5) = 82.5$

So $T(5) \approx 82.5° + 70° = 152.5°$

Then $\frac{dT}{dt} = 0.05(400° - 152.5°) \approx 12.375°$

So $\Delta T = 12.375° \cdot \Delta t = 12.375 \cdot (5) = 61.875$

Which makes

$$T(10) \approx 152.5° + 61.875° = 214.375°$$

7. A yam is removed from an oven and placed in outer space, where the temperature is –250°. If the yam starts out at 300°, and after 5 minutes it's dropped to 100°, how long will it be until it reaches the freezing point, 32°?

Use Newton's Law of Cooling: $\frac{dH}{dt} = k(-250° - H)$

Solve: $\int \frac{1}{-250° - H}\, dH = \int k\, dt$

$$-\ln|-250-H| = kt + C$$

$$\ln|-250-H| = -kt - C$$

$$|-250-H| = e^{-kt-C}$$

$$-H = Ae^{-kt} + 250$$

$H = -Ae^{-kt} - 250$ is a general solution.

Since $H(0) = 300°$:

$300 = -Ae^{-k(0)} - 250$

$550 = -A$

$A = -550$

So $H = 550e^{-kt} - 250$

Since $H(5) = 100°$:

$100 = 550e^{-k(5)} - 250$

$350 = 550e^{-5k}$

$\frac{7}{11} = e^{-5k}$

$-5k = \ln(7/11)$

$k = -\frac{\ln(7/11)}{5} \approx 0.09$

So $H = 550e^{-0.09t} - 250$

So after some time, 32°:

$32 = 550e^{-0.09t} - 250$

$\ln\left(\frac{282}{550}\right) = -0.09t$

$t \approx 7.39$ minutes

8. Biff is having trouble with differential equations. He says "Dude, there was this question on the review for our test, and it was like, whether a solution for $P' = kP(2 - P)$ can go through both (0,1) and (1,3). That's totally screwed up, isn't it? I mean, if they give you the initial things, you can always just work out the constants that make 'em happen, right? How can they even ask that?"

Explain to Biff why this question isn't as trivial as he thinks, and how you would answer it.

Biff, you need to recognize some things. That differential equation you have there is the formula for logistic growth, and it has an equilibrium at $P=2$. So 2 is the carrying capacity of that system, and if you start with a positive population less than that you'll increase up to 2, or if you start above 2 you'll drop to 2.

But your initial condition is below 2, and then later you want it to have gotten above 2. That can't happen, because the increase to a population of 2 is asymptotic - it never gets all the way there, let alone beyond it.

So just because it looks like a typical problem doesn't mean it'll go like usual. As long as the equation represents a situation where something's impossible, it won't be possible in the equations either.

9. We've been using the fact that if $f(t)$ and $g(t)$ are solutions to a differential equation of the form $ay'' + by' + cy = 0$ then $f(t) + g(t)$ will also be a solution. Prove it.

Well, if $f(t)$ is a solution to this equation that means:

$$a \cdot f''(t) + b \cdot f'(t) + c \cdot f(t) = 0$$

(which is just saying "it works in the equation").

And if $g(t)$ is a solution that means:

$$a \cdot g''(t) + b \cdot g'(t) + c \cdot g(t) = 0$$

But if both of those equations hold, then I can also say that the sum of the left-hand sides equals the sum of the right-hand sides:

$$\left[af''(t) + bf'(t) + cf(t)\right] + \left[ag''(t) + bg'(t) + cg(t)\right] = 0 + 0$$

And then re-arrange:

$$af''(t) + ag''(t) + bf'(t) + bg'(t) + cf(t) + cg(t) = 0$$

And factor:

$$a\left(f''(t) + g''(t)\right) + b\left(f'(t) + g'(t)\right) + c\left(f(t) + g(t)\right) = 0$$

But that's exactly what it means to say that the function $f(t) + g(t)$ is a solution to the differential equation. $\square$

10. Consider the differential equation $y'' - 2y' = 2y/t$. It's not quite the form that we can use a characteristic polynomial on, but something similar works: Try to find a solution of the form $y = te^{at}$ for some constant a.

Well, if $y = t \cdot e^{at}$ then

$$y' = 1 \cdot e^{at} + t \cdot a \cdot e^{at}$$

$$y'' = a \cdot e^{at} + a \cdot e^{at} + t \cdot a \cdot a \cdot e^{at}$$

So I'll substitute that in and see what happens:

$$(a \cdot e^{at} + a \cdot e^{at} + ta^2 e^{at}) - 2(e^{at} + tae^{at}) = \frac{2(te^{at})}{t}$$

$$2ae^{at} + ta^2 e^{at} - 2e^{at} - 2tae^{at} = 2e^{at}$$

And with some clever factoring: $(2a-2)e^{at} + (a^2 - 2a)te^{at} = 2e^{at}$

So there are $2a-2$ of those e^{at} pieces on the left, and 2 of them on the right, so

$$2a - 2 = 2$$

$$2a = 4$$

$$a = 2$$

I'll also make sure this works for the $a^2 - 2a$ of those te^{at} pieces on the left, and none of them on the right:

$$a^2 - 2a = 0$$

$$a(a-2) = 0$$

$$a = 0 \text{ or } a = 2$$

Yep. So $y = te^{2t}$ is a solution.