Exam 3 Differential Equations 4/9/04

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Give an example of eigenvalues that would produce a saddle at the origin.

a saddle occurs when $\lambda_1 < 0 < \lambda_2$

so

$\lambda_1 = -1$

$\lambda_2 = 1$ will work.

Great

2. If you know that a planar system has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 0$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, write a general solution for the system.

$$Y(t) = e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{0t}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

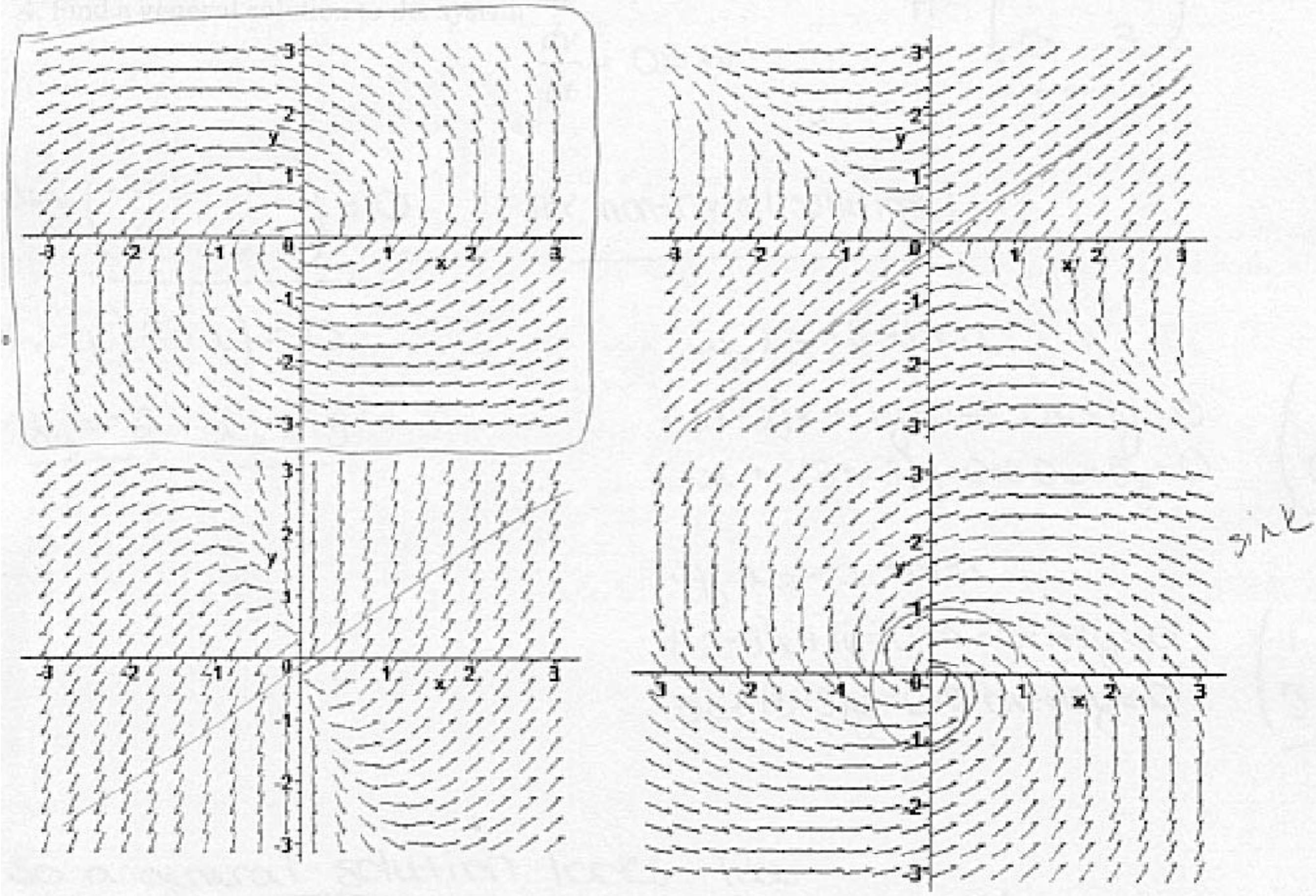
$$Y(t) = e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} \leftarrow \text{specific sol.}$$

to make general add coeff.

$$\boxed{Y(t) = k_1 e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2\begin{pmatrix} 3 \\ 5 \end{pmatrix}}$$

Excellent

3. Which of the phase planes shown below could correspond to a planar system with eigenvalues $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$?



For complex lamda values w/ a real part, we must analyze as follows:

$\lambda = \alpha \pm \beta i$

* $\alpha > 0$ so it's a spiral source
* $\beta = 1$

Excellent

4. Find a general solution to the system $\dfrac{dx}{dt} = -2x + y$, $\dfrac{dy}{dt} = 0x + 3y$

$A = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$

* $\det\begin{pmatrix} -2-\lambda & 1 \\ 0 & 3-\lambda \end{pmatrix} = 0$ (* for nontrivial solutions)

$(-2-\lambda)(3-\lambda) = 0$

$\lambda_1 = -2 \quad \lambda_2 = 3$

if $\lambda_1 = -2$ then

$(-2+2)x + 1y = 0 \Rightarrow 0x + 1y = 0$

$0x + (3+2)y = 0 \Rightarrow 0x + 5y = 0$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

if $\lambda_2 = 3$ then

$(-2-3)x + 1y = 0 \Rightarrow -5x + 1y = 0$

$0x + (3-3)y = 0 \Rightarrow 0x + 0y = 0$

$\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

so a general solution looks like

$$\vec{Y}(t) = k_1 e^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Well done

5. The planar system $\begin{aligned}\frac{dx}{dt} &= 2x-3y \\ \frac{dy}{dt} &= 3x+2y\end{aligned}$ has eigenvalues $\lambda_1 = 2+3i$ and $\lambda_2 = 2-3i$, and corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$. Write a general solution to this system in a form not involving imaginary exponents.

so for $\lambda_1 = 2+3i$:

$$Y(t) = k_1 e^{(2+3i)t}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= k_1 e^{2t} e^{3it}\begin{pmatrix} 1 \\ -i \end{pmatrix} = e^{2t}(\cos 3t + i\sin 3t)\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$= \begin{pmatrix} e^{2t}\cos 3t + ie^{2t}\sin 3t \\ -ie^{2t}\cos 3t + e^{2t}\sin 3t \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + ie^{2t}\begin{pmatrix} \sin 3t \\ -\cos 3t \end{pmatrix}$$

Excellent

general form:

$$\boxed{Y(t) = k_1 e^{2t}\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} \sin 3t \\ -\cos 3t \end{pmatrix}}$$

6. For which values of the constant b will the system of differential equations $\begin{aligned}\frac{dx}{dt} &= 2x + by \\ \frac{dy}{dt} &= 4x + 6y\end{aligned}$ have solutions involving sine and cosine?

$$\det\begin{pmatrix} 2-\lambda & b \\ 4 & 6-\lambda \end{pmatrix} = 0 = (2-\lambda)(6-\lambda) - 4b = 12 - 8\lambda + \lambda^2 - 4b$$

$$0 = \lambda^2 - 8\lambda + (12-4b)$$

$a = 1$

$b = -8$

$c = (12-4b)$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{8 \pm \sqrt{64 - 4(1)(12-4b)}}{2(1)}$$

To yield sines and cosines in the solution the eigenvalues must be complex. In order to obtain this the sum under the radical must be less than zero

$$= \frac{8 \pm \sqrt{64 - 48 + 16b}}{2} = \frac{8 \pm \sqrt{16 + 16b}}{2} = \frac{8 \pm \sqrt{16(1+b)}}{2} = \frac{8 \pm 4\sqrt{1+b}}{2}$$

$$= 4 \pm 2\sqrt{1+b}$$

so $1 + b < 0$

$b < -1$

Nice Job

7. Find a particular solution to the planar system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\mathbf{Y}$ which satisfies the initial condition $\mathbf{Y}_0 = (1,0)$.

$\det\begin{pmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{pmatrix} = 0$ (for nontrivial solutions!)

$(-2-\lambda)(-4-\lambda) + 1 = 0$

$8 + 4\lambda + 2\lambda + \lambda^2 + 1 = 0$

$\lambda^2 + 6\lambda + 9 = 0$

$(\lambda+3)(\lambda+3) = 0$

$\lambda = -3$ ← repeated eigenvalues!

* for repeated eigenvalues, we must use the solution form of $\vec{Y}(t) = e^{\lambda t}\vec{V}_0 + te^{\lambda t}\vec{V}_1$, where $\vec{V}_1 = (A - \lambda I) * \vec{V}_0$

compute $\vec{V}_1$: $\vec{V}_1 = \left[\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} + \begin{pmatrix} +3 & 0 \\ 0 & +3 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \underset{2\times2}{\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}} \underset{2\times1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\vec{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

So the solution is:

$$\vec{Y}(t) = e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Wonderful

8. Suppose that the planar system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -0.2 & 0.1 \\ 0.0 & -0.1 \end{pmatrix} \mathbf{Y}$ represents the populations of two species of fish, creatively labeled as "species X" and "species Y", where x(t) actually gives the number of fish of species X in a pond above or below the natural equilibrium population and y(t) gives the number of fish of species Y present in the pond.

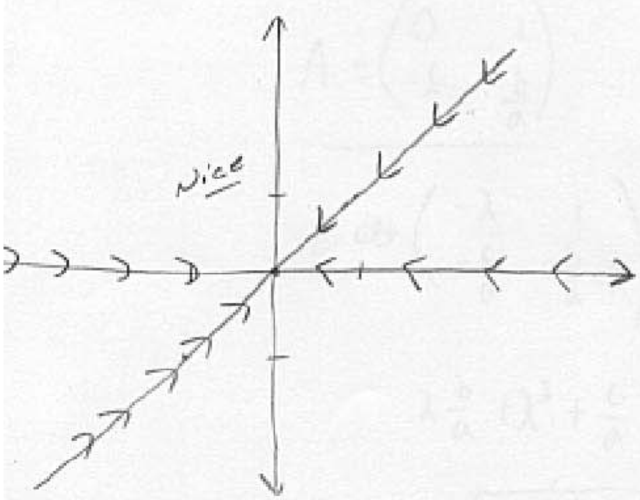
$x(t) = x - x_0$ $\quad\quad\quad$ $y(t) = y(t)$

The eigenvalues of the system are $\lambda_1 = -0.2$ and $\lambda_2 = -0.1$, with associated eigenvectors $\mathbf{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{V}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(a) Does the presence of fish of species Y benefit or hurt the population of species X?

(b) If all the fish of species Y are exterminated, what will happen to species X?

(c) If a small number of fish of species Y are introduced, with species X beginning at equilibrium, what will happen?



(a) $\frac{dx}{dt} = -0.2\,x(t) + 0.1\,y(t)$

it benefits the species X because the coefficient on y is positive

(b) $\frac{dx}{dt} = -0.2\,x(t)$

it will go to its own equilibrium x_0 because $x(t) = X - x_0$ so, if X is $> x_0$ the pop. will decrease to x_0 but if X is $< x_0$ the pop. will increase to x_0

Excellent

(c) The y population will decrease until it completely dies off, but the X pop. will increase while the Y species is present until it dies off and then the X species will fall back to equilibrium.

9. Convert the second-order differential equation $ay'' + by' + cy = 0$ to a system of first-order equations and find the equation in λ whose roots are the eigenvalues of the system.

Let $v = y'$

$\Rightarrow \quad av' + bv + cy = 0 \quad \Rightarrow \quad v' = \left(-\frac{b}{a}\right)v + \left(\frac{-c}{a}\right)y$

$y' = v$

$$A = \begin{pmatrix} -\frac{b}{a} & -\frac{c}{a} \\ 1 & 0 \end{pmatrix}$$

$$\det \begin{pmatrix} -\frac{b}{a} - \lambda & -\frac{c}{a} \\ 1 & -\lambda \end{pmatrix} = 0$$

Nice.

$$\lambda\left(\lambda + \frac{b}{a}\right) + \frac{c}{a} = 0$$

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \Rightarrow a\lambda^2 + b\lambda + c = 0$$

10. Jon is trying to make up a problem for a differential equations test. He wants a planar system with eigenvalues 2 and 3, with respective eigenvectors $\begin{pmatrix}1\\3\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. He knows that you can make a matrix with desired eigenvalues by putting the eigenvalues you want as entries on the main diagonal of the coefficient matrix with a zero in one of the other spots, but what coefficient matrix does he need in order to have his desired eigenvectors?

$\begin{pmatrix}2&0\\0&3\end{pmatrix}$ $\quad \begin{pmatrix}2-\lambda&0\\0&3-\lambda\end{pmatrix}$

$(2-\lambda)(3-\lambda)=0$

$\lambda=2 \qquad \lambda=3$

$\lambda=2: \quad 0x+1y=0 \qquad \begin{pmatrix}1\\0\end{pmatrix}$

$\lambda=3: \quad -1x+0y=0 \qquad \begin{pmatrix}0\\1\end{pmatrix}$

the solution to the system is: $Y(t)=e^{2t}\begin{pmatrix}1\\3\end{pmatrix}+e^{3t}\begin{pmatrix}0\\1\end{pmatrix}$

or $x(t)=e^{2t}$

$y(t)=3e^{2t}+e^{3t}$

so $x'(t)=2e^{2t}$

$y'(t)=6e^{2t}+3e^{3t}$

Nice approach!

<u>what coefficient on x & y will work?</u>

$x'(t)=2x+0y$? $\quad 2x-0y=2e^{2t}$ ✓

$y'(t)=3y-3x$? $\quad -3x+3y=-3e^{2t}+3(3e^{2t}+e^{3t})=-3e^{2t}+9e^{2t}+3e^{3t}$

$=6e^{2t}+3e^{3t}$ ✓

this works, so our coefficient matrix can be:

$\begin{pmatrix}2&0\\-3&3\end{pmatrix}$