Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate \( \int \sin t \, dt \).

\[-\cos t + C\]

Justification:
we know it's right \( b/c \) if we took the derivative of \(-\cos t\) we get \(\sin t\)

2. Suppose that you have an income source that pays you at a rate of $1000 per year right now, but will drop off steadily to $0 per year over the next 20 years. How much money will you receive from this source in total over the next 20 years?

\[ P = 1000 \]

If I had an equation I would take the integral, which means area under the curve, but since I have this graph, I'll just calculate it that way.

\[
\frac{(20 \times 1000)}{2} = 10,000
\]

\( \text{Slick!} \)
3. When the value of \( \int_{0}^{0.2} e^{-x^2} \, dx \) is approximated, Maple says that \( L_5 = 0.1981, R_5 = 0.1965, \) and \( M_5 = 0.1974. \) What will \( T_5 \) and \( S_5 \) be?

\[
T_5 \text{ is the average of } L_5 + R_5 = \frac{0.1981 + 0.1965}{2} = 0.1973
\]

\[
S_5 \text{ stands for Simpson's Rule which is } \frac{M_5 + 4M_1 + M_3}{3} = \frac{0.1973 + 0.1974 + 0.1974}{3} = 0.19737
\]

Simpson's rule gets us closest to the exact answer. 

4. Find the general antiderivative of \( f(t) = \frac{t}{1+5t^2} \) dt.

Let \( u = 1+5t^2 \) then \( \frac{du}{dt} = 10t \) and \( dt = \frac{du}{10t} \)

\[
\int \frac{t}{u} \, dt = \frac{1}{10} \int \frac{1}{u} \, du = \frac{1}{10} \ln |u| + C
\]

Wonderful!

\[
= \frac{1}{10} \ln |1+5t^2| + C
\]

I used \( u \) substitution to find the answer. To check I can take the derivative of my answer

\[
\ln |1+5t^2| = \frac{1}{1+5t^2} \cdot 10t, \text{ the ten cancels out by adding a } 10 \text{ and the } + \text{ moves to the top - and add } C.
\]
5. Evaluate \( \int xe^x \, dx \).

Integration by parts

\[ u = x \quad v = e^x \]
\[ u' = 1 \quad v' = e^x \]

\[ uv - \int v' \, u \, dx \]

I used int. by parts and the formula to check my answer. I find the deriv of my answer. Using the product rule:

\[ e^x + xe^x - e^x \]
gives me my original problem.

Then I added C.

Well done
6. Evaluate \( \int_3^\infty \frac{1}{x^2} \, dx \).

\[
\lim_{b \to \infty} \int_3^b \frac{1}{x^2} \, dx = \lim_{b \to \infty} \left[ -\frac{1}{x} \right]_3^b = \lim_{b \to \infty} \left( -\frac{1}{b} + \frac{1}{3} \right) = -\frac{1}{3}.
\]

Excellent. Your answer is correct. Just like the last question, you tell which answer is correct, given an indefinite integral. No matter what you got, its an 100% correct.
7. Biff is a calculus student at Anonymous State University, and he’s having some trouble with antiderivatives. He says “Dude, this sucks so bad! We’ve gotta take this calculus test on the computers, and it’s totally kicking my butt. It’s like, five antiderivative problems, and you’ve gotta get at least four right to pass, but it’s multiple choice from five answers, so the first days when I tried just guessing most of ‘em it went really bad. But you can only try it once each day, and I failed five times now, so I gotta get it tomorrow or I lose like twenty points every day. So the thing is, I heard these smart guys in the class talking, and they were laughing about how easy it was. They said something about just working the answers backwards and it only took a couple minutes, but I guess that’s something fancy that we haven’t learned in class yet. Do you know what they meant about backwards?”

Explain clearly to Biff how to tell which answer is correct, given an indefinite integral and a list of possible answers.

Hi Biff! First it is important for you to understand that antiderivatives are simply derivatives done backwards. If you take a function say \( y = x^2 \), find its derivative you’ll get \( \frac{dy}{dx} = 2x \). Now when you take the antiderivative of \( x^2 \), we get

\[
\int x^2 \, dx = \frac{1}{2} x^2 + C
\]

Look Biff, we got the same \( x^2 + C \) after we took its derivative & the antiderivative of the subsequent function. Your multiple choice test would have been a lot easier if you simply took the derivatives of the possible answers & see which one matches the question. Is the indefinite integral. I hope you now understand.

Excellent
8. Derive the formula \[
\int \frac{1}{(x-a)(x-b)} \, dx = \frac{1}{a-b} \left( \ln|x-a| - \ln|x-b| \right) + C. 
\]

To derive this we would use partial fractions. I wish
\[
\int \frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b} = 
\]

\[
A(x-b) + B(x-a) = 1 
\]

If \(x = a\) then \(A(a-b) + B(a-a) = 1\) \(A(a-b) = 1\) \(A = \frac{1}{a-b}\)

If \(x = b\) the \(A(b-b) + B(b-a) = 0 + B(b-a) = 1\)

\(B = \frac{1}{b-a}\)

\[
\frac{1}{a-b} \int \frac{1}{x-a} - \frac{1}{x-b} = 
\]

\[
\frac{1}{a-b} \left( \ln|x-a| - \ln|x-b| \right) + C
\]

**Excellent**

We would also take the derivative of this part, but it just equals one. So
\[
\int \frac{1}{x-a} = \ln(x-a)
\]
9. The integral \( \int_{0}^{R} \frac{R}{\sqrt{R^2 - x^2}} \, dx \) turns out to be important (I'll tell you why in the next chapter). Evaluate it.

\[
\int_{0}^{R} \frac{R}{\sqrt{R^2 - x^2}} \, dx = \lim_{b \to R} \int_{x=0}^{x=b} \frac{R}{\sqrt{R^2 - (R\sin \theta)^2}} \, d\theta
\]

\[
\frac{x}{R} = \sin \theta \Rightarrow \frac{dx}{d\theta} = R \cos \theta \Rightarrow dx = R \cos \theta \, d\theta
\]

\[
\int_{0}^{\arcsin \left( \frac{x}{R} \right)} \frac{1}{\sqrt{\cos^2 \theta}} \cdot R \cos \theta \, d\theta
\]

\[
= \lim_{b \to R} \left[ R \cdot \theta \right]_{x=0}^{x=b}
\]

\[
= \lim_{b \to R} \left[ R \cdot \arcsin \left( \frac{x}{R} \right) \right]_{0}^{b}
\]

\[
= \lim_{b \to R} \left( R \cdot \arcsin \left( \frac{b}{R} \right) - R \cdot \arcsin 0 \right)
\]

\[
= \arcsin 1
\]

\[
= R \cdot \frac{\pi}{2}
\]
10. a) Find the area below \( y = \sqrt{9 - x} \) but above the x axis and to the right of the y axis.

b) Find the value of \( b \) for which the area below \( y = \sqrt{b - x} \) but above the x axis and to the right of the y axis is equal to 9.

\[
\int_{a}^{b} \sqrt{b-x} \, dx = \int_{u=0}^{u=b} \left( \frac{2}{3} u^{\frac{3}{2}} \right) \, du
\]

Let \( u = 9 - x \), then \( du = -dx \)

\[
= -\frac{2}{3} b^{\frac{3}{2}} \left[ \frac{2}{3} \right]_{0}^{b}
\]

\[
= -\frac{2}{3} b^{\frac{3}{2}} - \frac{2}{3} b^{\frac{3}{2}}
\]

\[
= -\frac{4}{3} b^{\frac{3}{2}}
\]

So if we want the area to be 9, then set \( \frac{4}{3} b^{\frac{3}{2}} = 9 \) and solve...

\[
b^{\frac{3}{2}} = \frac{9}{4}
\]

\[
b = \left( \frac{27}{2} \right)^{\frac{2}{3}}
\]

\[
b = \frac{9}{\sqrt[3]{4}}
\]

This gives the area...