Exam 1b Calc 2 2/4/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Evaluate $\int \frac{1}{x^2}\,dx$. $= \int x^{-2}\,dx = -x^{-1} = \boxed{-\frac{1}{x}}$

justification

$\left(-\frac{1}{x}\right)' = (-x^{-1})' = x^{-2} = \frac{1}{x^2}$

Great

2. Suppose that you have an income source that pays you at a rate of \$2000 per year right now, but will drop off steadily to \$0 per year over the next 10 years. How much money will you receive from this source in total over the next 10 years?

$\int_0^{10} (2000 - 200t)\,dt$

$2000t - 100t^2 \Big|_0^{10}$

$2000(10) - 100(10)^2 - (2000(0) - 100(0)^2)$

\$10,000

Great

3. When the value of $\int_0^{0.5} \sin(x^2)\,dx$ is approximated, *Maple* says that $L_5 \approx 0.02728$, $R_5 \approx 0.05820$, and $M_5 \approx 0.04085$. What will T_5 and S_5 be, to five decimal places?

$T_5 = \frac{L_5 + R_5}{2} = \frac{0.02728 + 0.05820}{2} = 0.04274$

$S_5 = \frac{2M_5 + T_5}{3} = \frac{2(0.04085) + 0.04274}{3} = 0.04148$

Good

4. Evaluate $\int t^2 \cos\left(t^3\right) dt$.

$u = t^3$

$du = 3t^2\,dt$

$\frac{du}{3t^2} = dt$

$\int \cancel{t^2} \cos(u) \frac{du}{3\cancel{t^2}}$

$\frac{1}{3} \int \cos(u)\, du$

$\frac{1}{3} \cdot \sin(u) + C$

$\boxed{\frac{1}{3} \sin(t^3) + C}$

Nice!

5. Evaluate $\int_1^4 t \ln t \, dt$.

$u = \ln t \qquad v = \frac{t^2}{2}$

$u' = \frac{1}{t} \qquad v' = t$

$$\int_1^4 t \ln t \, dt$$

$$= \left[\ln t\left(\frac{t^2}{2}\right) - \int \frac{1}{t} \cdot \frac{t^2}{2}\, dt\right]_1^4$$

$$= \left[\frac{t^2}{2} \ln t - \frac{1}{2}\int t\, dt\right]_1^4$$

$$= \left[\frac{t^2}{2} \ln t - \frac{1}{2}\frac{t^2}{2}\right]_1^4$$

$$= \left(\frac{4^2}{2} \ln 4 - \frac{4^2}{4}\right) - \left(\frac{1}{2} \ln 1 - \frac{1}{4}\right)$$

$$= 8\ln 4 - 4 + \frac{1}{4}$$

$$= 8\ln 4 - \frac{15}{4}$$

Beautiful

6. Evaluate $\int \frac{z-1}{z^2+z}\,dz$.

$$= \int \frac{z-1}{z(z+1)}\,dz$$

I wish: $\frac{z-1}{z(z+1)} = \frac{A}{z} + \frac{B}{z+1}$

$$z-1 = A(z+1) + B(z)$$

if $z=0$: $(0)-1 = A(0+1) + B(0)$

$-1 = A(1) \rightarrow A = -1$

if $z=-1$: $(-1)-1 = A(-1+1) + B(-1)$

$-2 = B(-1) \rightarrow B = 2$

$$= \int \left(\frac{-1}{z} + \frac{2}{z+1}\right) dz$$

$$= 2\ln|z+1| - \ln|z| + C$$

Excellent

7. Biff is a calculus student at Anonymous State University, and he's having some trouble with antiderivatives. He says "Dude, I should be able to totally rock this calc exam, since I've got my frat's test files and this professor does almost the same test every single year. He changes stuff just enough that I have to kind of know some stuff, but it'll be so much easier than if I had to actually learn everything, you know? So the thing is, there's this one where I've got two different old tests, and there's different answers on this one integral problem but they're both marked right. Now obviously that means the slack-off grader, like, overlooked some mistake, right? Because one has a 'negative sine squared' answer, but the other has 'cosine squared' there. So one of them's gotta be wrong, but how do I figure out which?"

Explain clearly to Biff either how to tell which answer is correct (if they're really different) or how his two answers are equivalent.

$(-\sin^2 x)' = -2\sin x\cos x$

$(\cos^2 x)' = 2\cos x(-\sin x) = -2\sin x\cos x$

By taking the derivative of both answers, you find out that they are <u>equivalent</u>. To differentiate these, you have to use the chain rule to find that they both equal $-2\sin x\cos x$. And two things with the same derivative come from the same integral or anti derivative

↳ since $(-\sin^2 x)' = -2\sin x\cos x$
and $(\cos^2 x)' = -2\sin x\cos x$,

$\int -2\sin x\cos x\,dx$ can equal <u>either</u> $-\sin^2 x$ or $\cos^2 x$

<u>Yes.</u>

Improper!

8. Evaluate the integral $\int_0^1 \frac{1}{x^p}\,dx$. $= \lim_{a\to 0^+} \int_a^1 x^{-p}\,dx$

$$= \lim_{a\to 0^+} \left. \frac{x^{-p+1}}{-p+1} \right|_a^1$$

As long as $p \neq 1$ (see below)

$$= \lim_{a\to 0^+} \left(\frac{1^{1-p}}{1-p} - \frac{a^{1-p}}{1-p} \right)$$

$$= \lim_{a\to 0^+} \left(\frac{1}{1-p} - \frac{a^{1-p}}{1-p} \right)$$

Now from here it depends on p. If $1-p$ is positive, one thing happens, but if $1-p$ is negative then something else:

If $1-p>0$:

A small number to a positive power approaches zero.

$$= \frac{1}{1-p} - \frac{0}{1-p}$$

$$= \frac{1}{1-p}$$

If $1-p<0$:

A small number to a negative power is bad!

$$= \frac{1}{1-p} - \lim_{a\to 0^+} \frac{a^{1-p}}{1-p}$$

diverges

$$\therefore \int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \frac{1}{1-p} & \text{if } p<1 \\ \text{diverges} & \text{if } p \geq 1 \end{cases}$$

Notice that when $p=1$, the antiderivative is actually $\ln x$, which also diverges as x approaches 0^+.

9. The integral $\int_0^R \frac{R}{\sqrt{R^2 - x^2}}\,dx$ turns out to be important (I'll tell you why in the next chapter). ← Improper!

Evaluate it.

Trig. sub.!

let $x = R\sin\theta$

$\frac{dx}{d\theta} = R\cos\theta$

$dx = R\cos\theta \cdot d\theta$

$\frac{x}{R} = \sin\theta$

$\arcsin\left(\frac{x}{R}\right) = \theta$

$$= \lim_{b \to R} \int_0^b \frac{R}{\sqrt{R^2 - x^2}}\,dx$$

$$= \lim_{b \to R} \int_{x=0}^{x=b} \frac{R}{\sqrt{R^2 - (R\sin\theta)^2}} \cdot R\cos\theta \cdot d\theta$$

$$= \lim_{b \to R} \int_{x=0}^{x=b} \frac{\cancel{R}}{\cancel{R} \cdot \sqrt{1 - \sin^2\theta}} \cdot R\cos\theta\, d\theta$$

$$= \lim_{b \to R} \int_{x=0}^{x=b} \frac{R\cancel{\cos\theta}}{\cancel{\cos\theta}}\, d\theta$$

$$= \lim_{b \to R} R \cdot \theta \Big|_{x=0}^{x=b}$$

$$= \lim_{b \to R} R \cdot \arcsin\left(\frac{x}{R}\right) \Big|_0^b$$

$$= \lim_{b \to R} \left[R\arcsin\left(\frac{b}{R}\right) - R\arcsin\left(\frac{0}{R}\right)\right]$$

$$= \lim_{b \to R} R\arcsin\left(\frac{b}{R}\right)$$

$$= R\arcsin 1$$

$$= R \cdot \frac{\pi}{2}$$

10. A table of integrals larger than the one in our book includes the formula:

$$\int \frac{u\,du}{\sqrt{a+bu}} = \frac{2}{3b^2}(bu-2a)\sqrt{a+bu}+C.$$

Derive this formula.

$$\int \frac{u}{\sqrt{a+bu}}\,du$$

$$\boxed{\begin{array}{ll} U = u & V = \frac{2}{b}(a+bu)^{1/2} \\ U' = 1 & V' = (a+bu)^{-1/2} \end{array}}$$

$$\int \frac{u}{\sqrt{a+bu}}\,du = \frac{2u\sqrt{a+bu}}{b} - \int \frac{2}{b}(a+bu)^{1/2}\,du$$

$$= \frac{2u}{b}\sqrt{a+bu} - \frac{2}{b}\int (a+bu)^{1/2}\,du \longrightarrow \text{let } w = a+bu$$

$$\frac{dw}{du} = b \qquad du = \frac{dw}{b}$$

$$= \frac{2u}{b}\sqrt{a+bu} - \frac{2}{b}\int w^{1/2}\cdot\frac{dw}{b}$$

$$= \frac{2u}{b}\sqrt{a+bu} - \frac{2}{b^2}\left(\frac{2}{3}w^{3/2} + C\right)$$

$$= \frac{6bu}{3b^2}\sqrt{a+bu} - \frac{4}{3b^2}(a+bu)^{3/2} + C$$

$$= \frac{6bu}{3b^2}\sqrt{a+bu} - \frac{4}{3b^2}(a+bu)\sqrt{a+bu} + C$$

$$= \frac{6bu}{3b^2}\sqrt{a+bu} - \left(\frac{4a}{3b^2} + \frac{4bu}{3b^2}\right)\sqrt{a+bu} + C$$

$$= \sqrt{a+bu}\left(\frac{6bu}{3b^2} - \frac{4bu}{3b^2} - \frac{4a}{3b^2}\right) + C$$

$$= \sqrt{a+bu}\left(\frac{2bu}{3b^2} - \frac{4a}{3b^2}\right) + C$$

$$= \sqrt{a+bu}\cdot\frac{2}{3b^2}(bu-2a) + C$$

Marvelous!