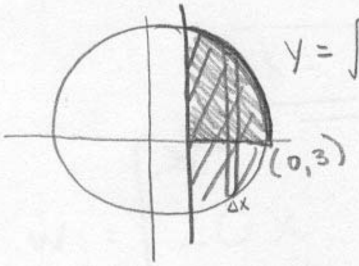
Exam 2a   Calc 2   2/25/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an integral for the area of the portion of the circle $x^2 + y^2 = 9$ which lies to the right of the line $x = 1$.



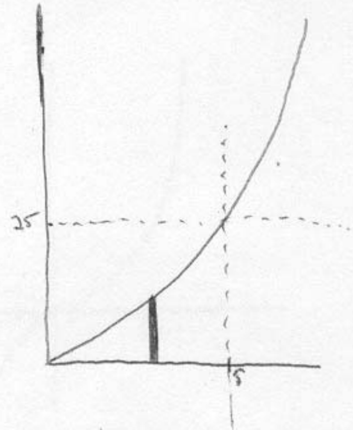
width of slice = Δx

height = $\sqrt{9 - x^2}$

$$\boxed{2\int_{1}^{3} \sqrt{9 - x^2}\, dx}$$

Good

2. Set up an integral for the volume of the solid obtained by rotating the region bounded by $y = 0$, $y = x^2$, and $x = 5$ around the x axis.



slice

radius = x^2

area = $\pi (x^2)^2$

volume = $\pi x^4 \Delta x$

$$\int_0^5 \pi x^4 \, dx$$

Yes

I hate the English system! Metric Rules! :)

3. A force of 10 pounds is required to hold a spring stretched 6 inches beyond its natural length. How much work, to the nearest tenth of a foot-pound, is done in stretching it from 4 inches to 12 inches beyond its natural length?

$F = 10$ lbs $\quad .5$ ft $\quad F = kx$ (for springs) $\quad 10\text{lbs} = k \cdot 0.5$ ft

$k = 20$ lbs/ft

$$W = \int_a^b F \,\Delta x$$

$$W = \int_{1/3}^{1} 20x \, dx$$

$$\boxed{8.9 \text{ ft} \cdot \text{lbs}}$$

$$W = 10x^2 \Big|_{1/3}^{1}$$

Well done

$$W = 10(1)^2 - 10(\tfrac{1}{3})^2$$

4. Set up an integral for the arc length of the portion of the curve $y = x^3$ between the points (-1,-1) and (2,8).

Arc Length $= \int_a^b \sqrt{1+[f'(x)]^2}\, dx \quad f(x) = x^3 \quad f'(x) = 3x^2$

The limits on the integral are just the x values of the coordinates

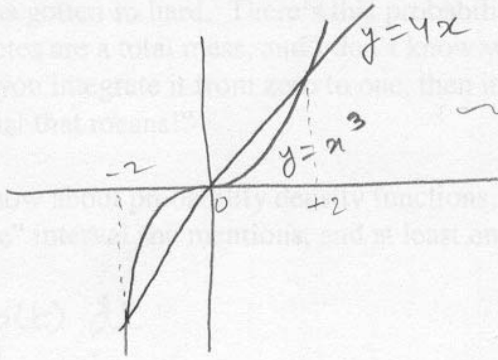
$$\text{Arc length} = \int_{-1}^{2} \sqrt{1+(3x^2)^2}\, dx$$

$$\boxed{= \int_{-1}^{2} \sqrt{1+9x^4}\, dx}$$

Excellent

5. Set up an integral or integrals for the x coordinate of the center of mass of the right-hand portion of the region bounded by $y = x^3$ and $y = 4x$.

$$\left.\begin{aligned} y &= x^3 \\ y &= 4x \end{aligned}\right\} \Rightarrow y = 4x - x^3 \rightarrow \boxed{y = 4x - x^3}$$



$x^3 - 4x = 0$

or, $x(x^2-4) = 0$

Either, $x = 0$

or, $x^2 - 4 = 0 \Rightarrow x^2 = 4 \therefore x = \pm 2$

$$\therefore \overline{X} = \frac{\int_0^2 x(4x - x^3)\,dx}{\int_0^2 (4x - x^3)\,dx}$$

Excellent

6. If $p(x) = \begin{cases} 0.3e^{-0.3x} & \text{if x} \geq 0 \\ 0 & \text{if x} < 0 \end{cases}$ is a probability density function giving the probability of a baby rhinoceros taking its first steps x days after birth, compute the median number of days after birth for a rhinoceros to take its first steps.

Find b for which:

$$\frac{1}{2} = \int_{-\infty}^{b} p(x)\,dx$$

$$\frac{1}{2} = \int_{-\infty}^{0} 0\,dx + \int_0^b 0.3e^{-0.3x}\,dx$$

$$\frac{1}{2} = 0 + -e^{-0.3x}\Big|_0^b$$

$$\frac{1}{2} = -\frac{1}{e^{+0.3b}} + \frac{1}{e^0}$$

$$\frac{1}{e^{0.3b}} = 1 - \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{e^{0.3b}}$$

$$e^{0.3b} = 2$$

$$0.3b = \ln 2$$

$$b = \frac{\ln 2}{0.3} \text{ days}$$

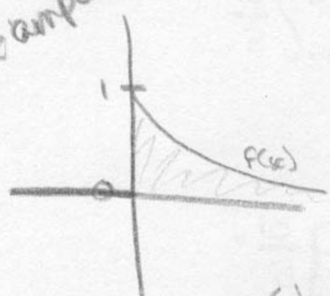
(≈ 2.31)

7. Bunny is a calculus student at Anonymous State University, and she's having some trouble with antiderivatives. Bunny says "Ohmygod, calculus has gotten *so* hard. There's this probability stuff, you know? And like, it's so confusing? My notes are a total mess, and I don't know what they mean at all. It's something like, maybe, like, if you integrate it from zero to one, then it's supposed to be infinite, I think. But I don't know what that means!"

Explain clearly to Bunny a couple of key things to know about probability density functions, including at least one thing involving the "zero to one" interval she mentions, and at least one thing involving infinity.

probability density functions need to meet some requirements. $\int_{-\infty}^{\infty} p(x)\,dx = 1$ so the infinity comes in not as the answer, but as the limits.
The "zero to one" thing, is the limits on the y-axis, which is the other requirement $0 \le p(x) \le 1$

example:



Perfect.

$$p(x) = \begin{cases} 0 & x < 0 \\ f(x) & x \ge 0 \end{cases}$$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

8. Congratulations, you've been handed the lucky exam! This exam is worth \$100, payable to the original recipient of this exam on this date 100 years from now (this offer does not extend to any individual other than the original recipient, and is not transferrable to any heirs or assignees). What constant income stream, payed out over the next 100 years and assuming continuous interest at 8%, would be of equivalent value?

$$\text{Future Value} = \int_0^M P(t)\, e^{r(M-t)}\, dt$$

$$100 = \int_0^{100} k \cdot e^{0.08(100-t)}\, dt$$

$$100 = k \int_0^{100} e^{8} \cdot e^{-0.08t}\, dt$$

$$\frac{100}{k} = e^{8} \cdot \frac{1}{-0.08} \cdot e^{-0.08t} \Big|_0^{100}$$

$$\frac{100}{k} = -\frac{e^{8}}{0.08} \cdot \left(e^{-8} - e^{0}\right)$$

$$\frac{100}{k} = -\frac{1}{0.08} + \frac{e^{8}}{0.08}$$

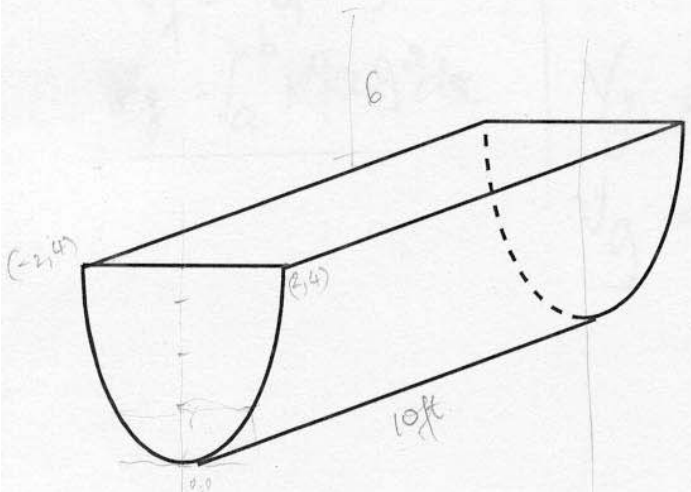
$$100 \cdot 0.08 = \left(e^{8} - 1\right) \cdot k$$

$$k = \frac{8}{e^{8} - 1}$$

$$k \approx \$0.0026846$$

or about 0.26 cents per year

9. Jon is planning to go into business selling new underground gasoline storage tanks to gas stations. The tanks will be shaped like the picture shown below, where the front and back faces are shaped like the parabola $y = x^2$ between (-2,4) and (2,4), and from front to back the tank measures 10 feet. Jon intends to market these tanks based on the claim that since more of the tank is near the surface than in a standard storage tank shaped like a cylinder lying on its side, the amount of work the pumps need to do will be less with these tanks than with conventional tanks. Set up an integral for the amount of work required to pump all the gasoline in one of Jon's tanks to a point 6 feet above the top of the tank (gasoline weighs 42 lb/ft^3).



$y = x^2$

$x = \pm\sqrt{y}$

Wonderful!

Width of a slice: $2\sqrt{y}$ (ft)

Area of a slice: $2\sqrt{y} \cdot 10$ (ft^2)

Volume of a slice: $20\sqrt{y}\ \Delta y$ (ft^3)

Weight of a slice: $20\sqrt{y}\ \Delta y$ (ft^3) $\cdot$ 42 lb/ft^3

Work to pump a slice $840\sqrt{y}\ \Delta y$ (lb) $\cdot (10-y)$ ft

$$\int_0^4 840\sqrt{y}(10-y)\,dy$$

10. Suppose that f(x) is a function (with f(x) $\geq$ 0 everywhere) whose length from $x = a$ to $x = b$ is given by L, and that the surface area of the solid obtained by rotating f(x) around the x axis is given by S. Let g(x) be another function, with g(x) = f(x) + 2. Express the surface area of g(x) in terms of the values L and S.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \qquad S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx$$

$$S.A._g = \int_a^b 2\pi g(x) \sqrt{1 + [g'(x)]^2}\, dx$$

Since $g(x) = f(x) + 2$, $g'(x) = f'(x)$

$$= \int_a^b 2\pi [f(x) + 2] \sqrt{1 + [f'(x)]^2}\, dx$$

$$= \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2}\, dx + \int_a^b 4\pi \sqrt{1 + [f'(x)]^2}\, dx$$

$$= S + 4\pi \cdot L$$

So the surface area of $g(x)$ is the surface area of $f(x)$ plus 4π times the length of $f(x)$!