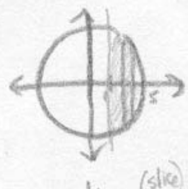
Exam 2b Calc 2 2/25/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Set up an integral for the area of the portion of the circle $x^2 + y^2 = 25$ which lies to the right of the line $x = 1$.

$y = \pm\sqrt{25-x^2}$



$$\int_1^5 2\sqrt{25-x^2}\,dx$$

radius (slice) $= 2\sqrt{25-x^2}$

area $= 2\sqrt{25-x^2}\ \Delta x$

Good

2. Set up an integral for the arc length of the portion of the curve $y = x^3$ between the points (-1,-1) and (2,8).

Arclength $= \int_a^b \sqrt{1+[f'(x)]^2}\,dx$

$y = x^3$

$y' = 3x^2$

$[y']^2 = 9x^4$

$$\int_{-1}^2 \sqrt{1+9x^4}\,dx$$

Good

FutureValue $= \int_0^m P(t)e^{r(m-t)}\,dt$

3. Write an integral for the future value of an income stream of $\boxed{\$2500}$ a year, for a period of $\boxed{6}$ years, if the continuous interest rate is $\boxed{7\%}$.

P(t) m r

$$\int_0^6 2500e^{0.07(6-t)}\,dt$$

Great!

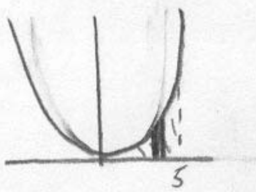
4. Set up an integral for the surface area of the solid obtained by rotating the region bounded by $y = 0$, $y = x^2$, and $x = 5$ around the x axis.

SA $= \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$

$f(x) = x^2$

$f'(x) = 2x$

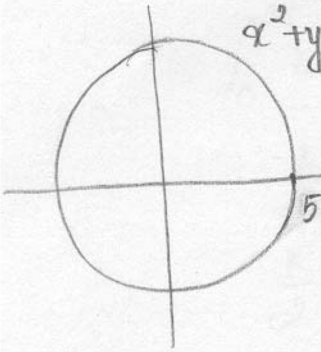
$f'(x)^2 = 4x^2$



$$\int_0^5 2\pi x^2\sqrt{1+4x^2}\,dx$$

Good

5. Write an integral or integrals for $\overline{x}$, the x coordinate of the center of mass of the first-~~octant~~ quadrant portion of a circle with radius 5.



$x^2+y^2=25$

$y=\sqrt{25-x^2}$

$$\overline{x} = \frac{\int_0^5 x\sqrt{25-x^2}\,dx}{\int_0^5 \sqrt{25-x^2}\,dx}$$

Nice

$12\,in = 1\,ft$

6. If a spring requires 10 foot-pounds of work to stretch it from a natural length of 24 inches to 30 inches, to what total length (to the nearest tenth of an inch) will 20 foot-pounds of work stretch it?

$$\int_0^{0.5} kx\,dx = 10$$

$$\frac{k}{2}x^2\Big]_0^{0.5} = 10$$

$$\frac{k}{2}(0.5)^2 = 10$$

$$\rightarrow k = 80$$

$$\int_0^a 80\,x\,dx = 20$$

$$40x^2\Big]_0^a = 20$$

$$40\,a^2 = 20$$

$$a^2 = \frac{1}{2}$$

$$a = 0.71 \text{ (foot)}$$

$$\rightarrow a = 8.5 \text{ inches}$$

Well done

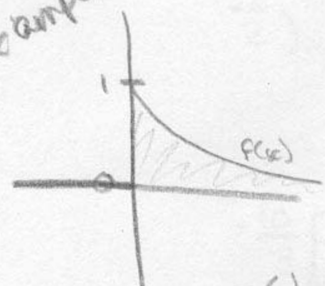
So, 20-foot-pounds of work with stretch the spring from its natural length 24 inches to 32.5 inches.

7. Bunny is a calculus student at Anonymous State University, and she's having some trouble with antiderivatives. Bunny says "Ohmygod, calculus has gotten *so* hard. There's this probability stuff, you know? And like, it's so confusing? My notes are a total mess, and I don't know what they mean at all. It's something like, maybe, like, if you integrate it from zero to one, then it's supposed to be infinite, I think. But I don't know what that means!"

Explain clearly to Bunny a couple of key things to know about probability density functions, including at least one thing involving the "zero to one" interval she mentions, and at least one thing involving infinity.

probability density functions need to meet some requirements. $\int_{-\infty}^{\infty} p(x)\,dx = 1$ so the infinity comes in not as the answer, but as the limits.
The "zero to one" thing is the limits on the y-axis, which is the other requirement $0 \le p(x) \le 1$

example



Perfect.

$$p(x) = \begin{cases} 0 & x < 0 \\ f(x) & x \ge 0 \end{cases}$$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

8. If $p(x)=\begin{cases} 0.3e^{-0.3x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ is a probability density function giving the probability of a baby rhinoceros taking its first steps *x* days after birth, compute the mean number of days after birth for a rhinoceros to take its first steps.

$$\bar{x} = \int_{-\infty}^{\infty} x \cdot p(x)\,dx$$

$$= \int_{-\infty}^{0} x \cdot 0\,dx + \int_{0}^{\infty} x \cdot 0.3e^{-0.3x}\,dx$$

$$= 0 + \lim_{b\to\infty} 0.3 \cdot \int_0^b x e^{-0.3x}\,dx \qquad \begin{matrix} u = x & v = \frac{-1}{0.3}e^{-0.3x} \\ u' = 1 & v' = e^{-0.3x} \end{matrix}$$

$$= \lim_{b\to\infty} 0.3\left(x \cdot \frac{-1}{0.3}e^{-0.3x} - \int \frac{-1}{0.3}e^{-0.3x}\,dx\right)\Big|_0^b$$

$$= \lim_{b\to\infty} \left(-xe^{-0.3x} + \int e^{-0.3x}\,dx\right)\Big|_0^b$$

$$= \lim_{b\to\infty} \left[\frac{-x}{e^{0.3x}} + \frac{-1}{0.3}e^{-0.3x}\right]_0^b$$

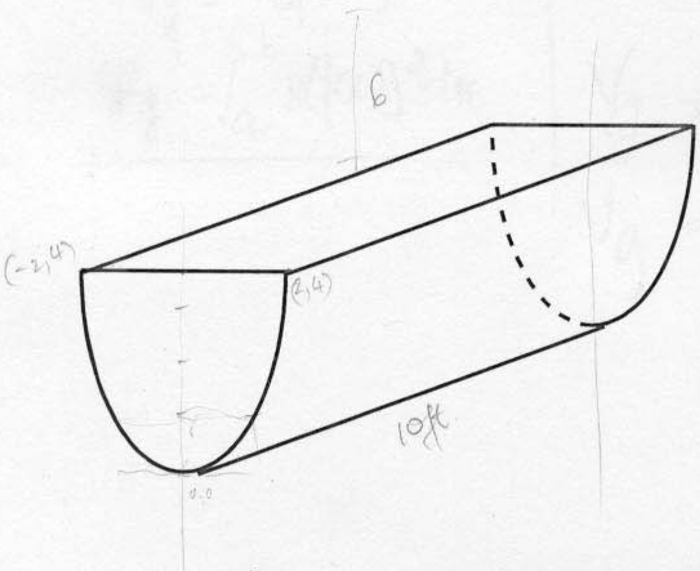
$$= \lim_{b\to\infty} \left(\frac{-b}{e^{0.3b}} - \frac{e^{-0.3b}}{0.3}\right) - \left(\frac{0}{e^0} - \frac{e^0}{0.3}\right)$$

$$= \lim_{b\to\infty} \left(\frac{-b}{e^{0.3b}} - \frac{1}{0.3e^{0.3b}} + \frac{1}{0.3}\right)$$

$$= \lim_{b\to\infty} \frac{-b}{e^{0.3b}} - \lim_{b\to\infty} \frac{1}{0.3e^{0.3b}} + \frac{10}{3}$$

$$\overset{L'H}{=} \lim_{b\to\infty} \frac{-1}{0.3e^{0.3b}} - 0 + \frac{10}{3} = 0 + \frac{10}{3} = \frac{10}{3} \text{ days}$$

9. Jon is planning to go into business selling new underground gasoline storage tanks to gas stations. The tanks will be shaped like the picture shown below, where the front and back faces are shaped like the parabola $y = x^2$ between (-2,4) and (2,4), and from front to back the tank measures 10 feet. Jon intends to market these tanks based on the claim that since more of the tank is near the surface than in a standard storage tank shaped like a cylinder lying on its side, the amount of work the pumps need to do will be less with these tanks than with conventional tanks. Set up an integral for the amount of work required to pump all the gasoline in one of Jon's tanks to a point 6 feet above the top of the tank (gasoline weighs 42 lb/ft^3).



$y = x^2$

$x = \pm\sqrt{y}$

Wonderful!

Width of a slice: $2\sqrt{y}$ (ft)

Area of a slice: $2\sqrt{y} \cdot 10$ (ft^2)

Volume of a slice: $20\sqrt{y}\ \Delta y$ (ft^3)

Weight of a slice: $20\sqrt{y}\ \Delta y$ (ft^3) $\cdot$ 42 lb/ft^3

Work to pump a slice $840\sqrt{y}\ \Delta y$ (lb)$\cdot(10-y)$ ft

$$\int_0^4 840\sqrt{y}(10-y)\,dy$$

10. Suppose that f(x) is a function, (with f(x) $\geq$ 0 everywhere) the area beneath which (but above the x axis) from $x = a$ to $x = b$ is given by A, and that the volume of the solid obtained by rotating the region beneath f(x) around the x axis is given by V. Let g(x) be another function, with g(x) = f(x) + 2. Express the volume of the region below g(x) between $x = a$ and $x = b$ in terms of the values A and V.

$$A = \int_a^b f(x)\,dx \qquad V = \int_a^b \pi\,[f(x)]^2\,dx$$

$$\begin{aligned}
V_g &= \int_a^b \pi\,(g(x))^2\,dx \\
&= \int_a^b \pi\,(f(x)+2)^2\,dx \\
&= \int_a^b \pi\left\{[f(x)]^2 + 4\cdot f(x) + 4\right\}dx \\
&= \int_a^b \pi\,[f(x)]^2\,dx + 4\pi\int_a^b f(x)\,dx + \int_a^b 4\pi\,dx \\
&= V + 4\pi\cdot A + 4\pi\,(b-a)
\end{aligned}$$

So the volume you get from rotating g(x) is the volume from rotating f(x), plus 4π times the area under f(x), plus 4π times the length of the interval.