Exam 3a Calc 2 4/1/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Write a 5[th] degree Taylor polynomial for the function f(x) = sin x centered at x = 0.

Since we decided to memorize sinx, I know it equals

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \boxed{x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{1}{6}x^3 + \frac{1}{120}x^5}$$

Great!

2. Find the sum of the series $\frac{2}{3} - \frac{2}{9} + \frac{2}{27} - \frac{2}{81} + \ldots$

Infinite geometric series $\frac{a}{1-r}$ $a = \frac{2}{3}$ $r = -\frac{1}{3}$ $|r|<1$? Yes

$$\frac{\frac{2}{3}}{1-(-\frac{1}{3})} = \frac{1}{2}$$

$\frac{4}{3}$

Excellent

3. Show that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges. $\frac{1}{1} + \frac{1}{2^3} + \frac{1}{3^3} \ldots$

I Know from the P series test that says $\frac{1}{n^p}$ converges if $p>1$ and diverges if $p<1$. Since $p=3$ and $3>1$, I Know it converges.

Nice!

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges

4. Use a 4^{th} degree Taylor polynomial to find an approximation of cos 0.2 to 8 decimal places.

$\cos(.2)$

We Know the polynomial for $\cos(x)$

$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

$\cos(.2) \approx = 1 - \frac{(.2)^2}{2} + \frac{(.2)^4}{4!}$

Excellent

$1 - .02 + \frac{.0016}{24} \approx .98006667$

$\boxed{\cos .2 \approx .98006667}$

5. Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$.

Using Ratio test,

$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$ becomes: $\lim_{n \to \infty} \left| \frac{x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{x^n} \right|$

$$= \lim_{n \to \infty} \left| \frac{\cancel{x^n} \cdot x}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{x^n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{x}{(2n+2)(2n+1)} \right|$$

Excellent

$$= 0$$

This means the Radius of convergence is ∞.
i.e. it converges for every value of x.

6. Biff is a calculus student at Anonymous State University, and he's having some trouble with series. Biff says "Man, this convergence stuff is kicking my ass. We just had our test over it and I guess everybody failed, 'cause the professor said we get to take it over again next week to make our grades better. So one of the questions was about whether approximations from Taylor polynomials are always accurate if you use a high enough degree polynomial. It was multiple choice, and I picked the answer that said 'yeah, as long as you don't have an arithmetic mistake', 'cause that's where I screw up a lot. But the machine scored it wrong, which I think is crap, because obviously it's true for me, right?"

Explain clearly to Biff why his answer is actually correct or not, and what sort of answer his professor likely counts as correct.

Poor Biff. Regardless of the degree to which you're taking your polynomial, as long as it's not in your interval of convergence, it will not be accurate. For any degree outside the interval of convergence, it will move in different directions. By relying on a higher degree, they do not become more accurate and this goes against what Biff is saying. The teacher mostlikely wanted an answer explaing that regardless of the degree, once it's outside the interval of convergence, it won't become anymore accurate.

Wonderful!

7. Students occasionally insist that $\sin^2 x$ is the same as $\sin(x^2)$. Show that the Taylor polynomials for these two functions are different.

I know: $\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}$

so substituting: $\sin(x^2) \approx x^2 - \frac{(x^2)^3}{6} + \frac{(x^2)^5}{120}$

or: $\sin(x^2) \approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120}$

but on the other hand:

$$\sin^2 x = (\sin x)(\sin x) \approx \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right)$$

$$\approx \underbrace{x^2 - \frac{x^4}{6} + \frac{x^6}{120} - \frac{x^4}{6}}_{} + \ldots$$

And right here we can tell they're different - $\sin^2 x$ has an x^4 term (all the rest of the product will have higher degree, so it can't cancel this out), whereas $\sin(x^2)$ jumped from x^2 to x^6.

8. Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series. positive Define a new series $\sum_{n=1}^{\infty} b_n$ by letting $b_n = \begin{cases} a_n & \text{if n is odd} \\ 0 & \text{if n is even} \end{cases}$. Does $\sum_{n=1}^{\infty} b_n$ converge?

1, 3

Since $\sum_{n=1}^{\infty} a_n$ converges

and we Know if n is odd b_n is equal to $\sum^{1} a_n$.

And if n is even b_n is $= 0$. ~~So we can use~~

the Meaning b_n is $\leq a_n$. We can use

the Comparison Test. $\sum_1 a_n$ & $\sum b_n$ are

both positive where b_n is $\leq a_n$. a_n converges

and since $b_n \leq a_n$ it converges as well.

Excellent!

9. Determine whether $x = {}^{-1}\!/_3$ is in the interval of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(3x)^n}{n^2+1}$.

If $x = {}^{-1}\!/_3$: $\displaystyle\sum_{n=0}^{\infty} \frac{(3 \cdot {}^{-1}\!/_3)^n}{n^2+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ Try A.S.T.!

✓ - This series alternates signs because of $(-1)^n$.

✓ - $\displaystyle\lim_{n\to\infty} \frac{1}{n^2+1} = 0$ because the numerator is constant and the denominator grows without bound.

✓ - It's decreasing because if $f(x) = \dfrac{1}{x^2+1}$

then $f'(x) = \dfrac{0 \cdot (x^2+1) - 1 \cdot 2x}{(x^2+1)^2}$

$= \dfrac{-2x}{(x^2+1)^2}$

and that's negative since (for positive x) the numerator is always negative and (as a square) the denominator is never negative. So with a negative denominator, it must be <u>decreasing</u>.

So it satisfies all three requirements for the Alternating Series Test and must be convergent, thus $x = {}^{-1}\!/_3$ <u>is</u> in the interval of convergence.

10. Use the following pieces of information to find a Taylor polynomial of degree 6 for the function $\cosh x$:

- $(\cosh x)' = \sinh x$
- The 5th degree Taylor polynomial for $\sinh x$ is $x + \frac{x^3}{6} + \frac{x^5}{120}$ (yes, all plus signs).
- $\cosh 0 = 1$.

Since: $\sinh x \approx x + \frac{x^3}{6} + \frac{x^5}{120}$

integrating: $\int \sinh x \, dx \approx \int \left(x + \frac{x^3}{6} + \frac{x^5}{120}\right) dx$

$$\cosh x \approx \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} + C \quad *$$

and when $x=0$: $\cosh 0 \approx 0 + 0 + 0 + C$

or: $1 \approx C$

So we take $C = 1$ and rewrite $*$ as:

$$\cosh x \approx 1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}$$