Exam 3b Calc 2 4/1/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Find, to the nearest thousandth, ,000 the first 4 partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.

$s_1 = 1 = \underline{1.000}$

$s_2 = 1 + \frac{1}{\sqrt{2}} = \underline{1.707}$

$s_3 = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \approx \underline{2.284}$ Great

$s_4 = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} \approx \underline{2.784}$

2. Find a 6th degree Taylor polynomial for the function f(x) = $e^{(-x^2)}$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$e^{(-x^2)} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \ldots$$

$$e^{(-x^2)} \approx 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}$$

Great

3. Show that $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Use the integral test:

$$\int_1^{\infty} \frac{1}{x^3}dx = \lim_{b \to \infty} \int_1^b x^{-3}\,dx$$

$$= \lim_{b \to \infty} \frac{x^{-2}}{-2} = \lim_{b \to \infty} -\frac{1}{2x^2}\Big]_1^b$$

$$= \lim_{b \to \infty} -\frac{1}{2b^2} + \frac{1}{2} = \frac{1}{2} \quad \text{converges}$$

So $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the integral test Excellent!

4. Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{3^n}$.

Use the ratio test:

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{3^{n+1}} \cdot \frac{3^n}{x^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x}{3} \right| = \left| \frac{x}{3} \right|$$

If $\left| \frac{x}{3} \right| < 1$, it converges.

So $|x| < 3$

Radius of convergence is 3.

Great

5. Use a power series of degree at least 8 to approximate the value of $\int_0^{0.5} \frac{1}{1+x^6}\,dx$ to 8 decimal places.

I know $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

so $\frac{1}{1-(-x^6)} \approx 1 - x^6 + (-x^6)^2$

or $\frac{1}{1+x^6} \approx 1 - x^6 + x^{12}$

Then $\int_0^{0.5} \frac{1}{1+x^6}\,dx \approx \int_0^{0.5} (1 - x^6 + x^{12})\,dx$

$$\approx \left[x - \frac{x^7}{7} + \frac{x^{13}}{13} \right]_0^{0.5}$$

$$\approx 0.5 - \frac{0.5^7}{7} + \frac{0.5^{13}}{13}$$

$$\approx 0.49889332$$

6. Biff is a calculus student at Anonymous State University, and he's having some trouble with series. Biff says "Man, this convergence stuff is kicking my ass. We just had our test over it and I guess everybody failed, 'cause the professor said we get to take it over again next week to make our grades better. So one of the questions was about whether approximations from Taylor polynomials are always accurate if you use a high enough degree polynomial. I picked the answer that said 'yeah, as long as you don't have an arithmetic mistake', 'cause that's where I screw up a lot. But the machine scored it wrong, which I think is crap, because obviously it's true for me, right?"

Explain clearly to Biff why his answer is actually correct or not, and what sort of answer his professor likely counts as correct.

It all depends on the series, Biff. Sometimes, such as with the T.P. of $\ln|1-x|$ the function has an asymtote. Since polynomials do not have asymtotes a higher degree polynomial will not help and will actually hurt after you have past the Polynomials radius of convergence.

The professor would have most likely accept is a higher degree polynomial will get you a better approximation of its function as long as its x value is within the radius of convergence.

Wonderful Answer!

7. a) Find the sum of the series $\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\frac{1}{5^4}+\ldots$

b) For what value of b is the sum of the series $\frac{1}{b}+\frac{1}{b^2}+\frac{1}{b^3}+\frac{1}{b^4}+\ldots$ equal to ½?

a) this is a <u>geometric series</u>.

$a = \frac{1}{5}$

$r = \frac{1}{5}$

$S = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{5}\cdot\frac{5}{4} = \frac{1}{4}$

b). $a = \frac{1}{b}$

$r = \frac{1}{b}$

$S = \frac{\frac{1}{b}}{1-\frac{1}{b}}$

$\frac{1}{2} = \frac{\frac{1}{b}}{\frac{b-1}{b}}$

$\frac{1}{2} = \frac{1}{b}\cdot\frac{b}{b-1}$

$\frac{1}{2} = \frac{1}{b-1}$

$b-1 = 2$

$b = 3.$

So if $b=3$, sum of the series $\frac{1}{b}+\frac{1}{b^2}+\frac{1}{b^3}+\frac{1}{b^4}+\cdots = \frac{1}{2}$

Nice Job!

8. A trig identity sometimes used in integration problems says that 2 sin x cos x = sin $2x$. Verify that the 3[rd] degree Taylor polynomials for both sides of this identity match.

* $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\approx x - \frac{x^3}{3!}$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

$\approx 1 - \frac{x^2}{2}$

$\sin x \cos x \approx \left(x - \frac{x^3}{3!}\right)\left(1 - \frac{x^2}{2}\right)$

$\approx x - \frac{x^3}{2} - \frac{x^3}{6}$

$\approx x - \frac{2}{3}x^3$

$2 \sin x \cos x \approx 2x - \frac{4}{3}x^3$ (1)

* $\sin x \approx x - \frac{x^3}{6}$

$\sin 2x \approx 2x - \frac{(2x)^3}{6}$

$\approx 2x - \frac{8x^3}{6}$

$\approx 2x - \frac{4}{3}x^3$ (2)

Beautiful

From (1); (2) we have $2 \sin x \cos x = \sin 2x$

9. Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive terms. Define a new series $\sum_{n=1}^{\infty} b_n$ by letting $b_n = \begin{cases} a_n & \text{if n is odd} \\ 2a_n & \text{if n is even} \end{cases}$. Does $\sum_{n=1}^{\infty} b_n$ converge?

Comparison!

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$$

vs.

$$a_1 + 2a_2 + a_3 + 2a_4 + a_5 + 2a_6 + \cdots$$

Well, since I know $\sum_{n=1}^{\infty} a_n$ converges, then $2 \cdot \sum_{n=1}^{\infty} a_n$ converges, or $\sum_{n=1}^{\infty} 2a_n$ converges. And each term in this series is greater than or equal to the corresponding terms in $\sum_{n=1}^{\infty} b_n$, since for odd terms $b_n = a_n \le 2a_n$ and for odd terms $b_n = 2a_n \le 2a_n$. Then by the comparison test, $\sum_{n=1}^{\infty} b_n$ is smaller than a series that converges (and yet non-negative) so $\sum_{n=1}^{\infty} b_n$ must also converge.

10. Determine whether $x = -1/a$ is in the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(ax)^n}{n^2}$.

$x = -1/a$

$$\sum_{n=1}^{\infty} \frac{\left(a \cdot \frac{-1}{a}\right)^n}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Use the alternating series test:

- the sign alternate: $(-1)^n$.
- $\lim_{n\to\infty} \frac{1}{n^2} = 0$
- $f(x) = \frac{1}{x^2}$; $f'(x) = -\frac{2}{x^3}$ (negative)

So the sequence $\frac{1}{n^2}$ is decreasing

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges.

$x = -\frac{1}{a}$ is in interval of convergence

Excellent!