Exam 4a Calc 2 4/22/2005

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Circle all of the candidates below which **are** $\underbrace{\text{differential equations}}_{\text{a.k.a. "equations with derivatives"}}$:

a) $\dfrac{dp}{dt} = kp(500 - p)$

b) $5x = 25$ ← No derivative!

c) $y = Ae^{kt}$ ← No derivative, though it's a solution to some diff. eq's

d) $y'' + 7y' + 12y = 0$

e) $v' = -32.2$

f) $k\dfrac{dy}{dx} + 7y - \ln|500 - t| + C$ ← Not an equation!

2. Determine whether $y = 2 + 3e^{2x+4}$ is a solution to the differential equation $\dfrac{dy}{dx} = 2y - 4$.

$y' = 3e^{2x+4}(2)$

$y' = 2y - 4$

$y' = 6e^{2x+4}$

$6e^{2x+4} \overset{?}{=} 2(2 + 3e^{2x+4}) - 4$

$6e^{2x+4} \overset{?}{=} 4 + 6e^{2x+4} - 4$

$6e^{2x+4} \overset{?}{=} 6e^{2x+4}$

Great

Yes, it is a solution

3. Give an example of a differential equation for a population undergoing exponential growth.

$$\boxed{\frac{dP}{dt} = KP}$$

the change in pop. over time is proportional to the current pop. times a constant.

Great

4. Find a general solution to the differential equation $y'' - 4y' - 5y = 0$.

let the general solⁿ ~~by~~ be $y = e^{st}$.

$\therefore y' = se^{st}$

$\therefore y'' = s^2 e^{st}$

$\therefore y'' - 4y' - 5y = 0$

or, $s^2 e^{st} - 4se^{st} - 5e^{st} = 0$

or, $e^{st}(s^2 - 4s - 5) = 0$

Since e^{st} can't be 0; $s^2 - 4s - 5$ may be zero.

$\therefore s^2 - 4s - 5 = 0$

or, $s^2 - 5s + s - 5 = 0$

or, $s(s-5) + 1(s-5) = 0$

or, $(s-5)(s+1) = 0$

$\therefore$ either $s = 5$ or, $s = -1$

$\therefore$ The general solⁿ is ; $y(t) = Ae^{5t} + Be^{-t}$

Excellent

5. Find a general solution to the differential equation $\frac{dQ}{dt} - \frac{Q}{k} = 0$, where k is a constant.

$\frac{dQ}{dt} = \frac{Q}{k}$ Separation of variables!

$\int \frac{1}{Q} \cdot dQ = \int \frac{1}{k} \cdot dt$

$\ln|Q| = \frac{1}{k} \cdot t + C$

where this constant absorbs the absolute value bars.

$Q = A \cdot e^{t/k}$

6. Suppose that the population of fish (measured in thousands) in a large lake is governed by

$$\frac{dp}{dt} = 0.004p(30 - p) - 1$$

and $p(0) = 25$. Use Euler's method with $\Delta t = 10$ to approximate the population of fish in the lake 30 years from now to the nearest hundred.

t	p	Δp
0	25	-5
10	20	-2
20	18	-1.36
30	16.64	

$$\frac{dp}{10} = .004(25)(30-25) - 1 = -.5$$

$$\frac{dp}{10} = .004(20)(30-20) - 1 = -.2$$

$$\frac{dp}{10} = .004(18)(30-18) - 1 = -.136$$

Excellent

Population of fish in the lake in 30 yrs will be 16,600 fish.

7. Bunny is a calculus student at Anonymous State University, and she's having some trouble with differential equations. Bunny says "OHmygod, these are so confusing. We had this problem on our problem set where, like, we did the oily method, and then it asked about if there was an equilibrium, right? So we did the oily part, and it was totally obvious that it was going up slower and slower, right? So I said there wasn't an equilibrium, because it would never actually get there. The grader gave me, like, almost no points for that second part, which is totally wrong, because he said the first part was totally right. So if it doesn't get there, then it's not an equilibrium, right?

Explain clearly to Bunny whether her statements about equilibrium are correct, or if some refinement is in order.

It sounds like a logistic growth problem Bunny. The graph might resemble

If your equation is of the form $\frac{dP}{dt} = KP(200-P)$ where 200 is some random number I made up, to find its equilibrium, simply put 0 in for $\frac{dP}{dt}$ and solve. The answer will give you info about your horrizontal asymptote. That is your equilibrium pt. and your graph will never get there. You might ask what does "equilibrium" mean? where you have an equil. pt. is where the rate of change is zero. If your equation started at the equil. pt. there would be no change and your graph would be a horrizontal line. Though your first answer was right, you need to know that there was an equilibrium point.

Wonderful Answer

8. In some chemical reactions, the rate at which the amount of a substance changes with time is proportional to the amount present. For example, this is the case as δ-glucono-lactone acid changes into gluconic acid.[1]

a) Write a differential equation satisfied by y, the quantity of δ-glucono-lactone present at time t.

b) If 100 grams of δ-glucono-lactone is reduced to 54.9 grams in one hour, how many grams will remain after 3 hours?

a) $\frac{dg}{dt} = Kg$ g: amt glucono present

$.549 = K\cdot$

b) Separation of variables! :)

$\frac{dg}{g} = \frac{Kg}{g}dt$

$\int \frac{dg}{g} = \int K dt$

$\ln|g| = Kt + C$

$e^{Kt+C} = g$

general Solution: Ae^{Kt} = amt. of glucono

amt glucose = Ae^{Kt}

$54.9 = 100e^{K(1)}$

$\ln|.549| = \ln|e^K|$

$K = \ln|.549|$

$K = -.59965$

amt glucose = $100e^{-.59965(3)}$

$\boxed{16.5 \text{ grams}}$

Well done!

[1]Borrowed from Hughes-Hallett et al, 3rd ed., p. 512.

9. Small raindrops' velocities can be modeled with the differential equation

$$\frac{dv}{dt} = -32.2 + \frac{4.60\times10^{-4}}{d^2}v^2$$

where d is the diameter of the raindrop in feet.

a) Find the equilibrium velocity for a raindrop with diameter $d = 1.00 \times 10^{-2}$ feet, expressing your answer in scientific notation correct to 3 significant digits.

b) Find a general solution to this differential equation.

a) To find equilibrium, set $\frac{dv}{dt}$ equal to zero:

$$0 = -32.2 + \frac{4.60 \cdot 10^{-4}}{(1.00 \cdot 10^{-2})^2} \cdot v^2$$

$$32.2 = 4.6 v^2$$

$$7 = v^2$$

$v \approx 2.6457$ feet/sec, or about 2.65×10^0 feet/sec.

b) $$\frac{dv}{dt} = -32.2 + 4.6v^2$$

$$\int \frac{dv}{-32.2+4.6v^2} = \int dt$$

$$\frac{1}{4.6}\int \frac{dv}{v^2-7} = t + C$$

$$\int \frac{dv}{(v+\sqrt{7})(v-\sqrt{7})} = 4.6t + D$$

$$\frac{1}{-\sqrt{7}-\sqrt{7}}\left(\ln|v+\sqrt{7}| - \ln|v-\sqrt{7}|\right) = 4.6t + E$$

$$\ln\left|\frac{v+\sqrt{7}}{v-\sqrt{7}}\right| = -2\sqrt{7} \cdot 4.6t + F$$ is an implicit solution.

10. We used characteristic polynomials to solve differential equations of the form

$$ay'' + by' + cy = 0$$

but a variation of that approach can be used to solve equations like

$$y'' - 4y' - 5y = \sin x.$$

Find a solution to this equation by guessing that probably there's a solution of the form

$$y = A\sin x + B\cos x$$

for some values of the coefficients A and B, and proceeding to find suitable values for A and B.

If $y = A\sin x + B\cos x$

then $y' = A\cos x - B\sin x$

and $y'' = -A\sin x - B\cos x$

So to be a solution:

$(-A\sin x - B\cos x) - 4(A\cos x - B\sin x) - 5(A\sin x + B\cos x) = \sin x$

or, collecting $\sin x$ and $\cos x$ terms:

$(-A\sin x + 4B\sin x - 5A\sin x) + (-B\cos x - 4A\cos x - 5B\cos x) = \sin x$

But for these functions to be equal, the $\sin x$ coefficients would have to match and the $\cos x$ coefficients too, so:

$$\left.\begin{array}{l} -A + 4B - 5A = 1 \\ -B - 4A - 5B = 0 \end{array}\right\} \text{Hey! It's a system of two equations!}$$

or $-6A + 4B = 1$

$-4A - 6B = 0$

but this second equation says

$A = \frac{-3}{2}B$

and substituting this into

$-6A + 4B = 1$

gives $-6\left(\frac{-3}{2}B\right) + 4B = 1$

or $+9B + 4B = 1$

$13B = 1$

$B = \frac{1}{13}$

and putting this back into

$A = \frac{-3}{2}B$

gives $A = \frac{-3}{2}\left(\frac{+1}{13}\right) = \frac{-3}{26}$

So $y = \frac{-3}{26}\sin x + \frac{1}{13}\cos x$ is a solution!