Exam 4b **Calc 2** **4/22/2005**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State the definition of a differential equation.

A differential equation is any equation with a derivative in it.

Yes!

2. Determine whether $y = 2 + 3e^{2x-4}$ is a solution to the differential equation $\frac{dy}{dx} = 2y - 4$.

$$y' = 3e^{2x-4}(2)$$
$$= 6e^{2x-4}$$

$$(6e^{2x-4}) \stackrel{?}{=} 2(2+3e^{2x-4}) - 4$$
$$6e^{2x-4} \stackrel{?}{=} \cancel{4} + 6e^{2x-4} \cancel{-4}$$
$$6e^{2x-4} = 6e^{2x-4}$$

Good

yes, it is a solution

3. Give an example of a differential equation for a population undergoing logistic growth with a carrying capacity of 8000.

$$\frac{dP}{dt} = K \cdot P\,(C - P)$$

$$\boxed{\frac{dP}{dt} = KP(8000 - P)}$$

Yup.

4. Find a general solution to the differential equation $y'' + 4y' - 5y = 0$.

Probably $y = e^{st}$

$y' = se^{st}$

$y'' = s^2 e^{st}$

$(s^2 e^{st}) + 4(se^{st}) - 5(e^{st}) = 0$

$e^{st}(s^2 + 4s - 5) = 0$

$e^{st}(s-1)(s+5) = 0$

$s = 1$ or $s = -5$

so $y = Ae^{t} + Be^{-5t}$

Great!

5. Find a general solution to the differential equation $\frac{dP}{dt} - aP = b$, where a and b are constants.

$$\frac{dP}{dt} - ap = b$$

$$\frac{dP}{dt} = b + ap$$

$$dP = (b+aP)\,dt$$

let $u = b + ap$

$du/dp = a$

$du/a = dp$

$$\int \frac{1}{b+ap}\,dP = \int dt$$

$$\int \frac{1}{u}\,\frac{du}{a} = \int dt$$

$$\frac{1}{a}\int \frac{1}{u}\,du = \int dt$$

$$\frac{1}{a}\ln|u| = t + C$$

$$\ln|u| = at + C$$

$$\cancel{e}^{\ln|u|} = e^{at+C}$$

$$|u| = e^{at} \cdot e^{C}$$

$$u = Ae^{at}$$

$$b + ap = Ae^{at}$$

$$ap = Ae^{at} - b$$

$$\boxed{p = \frac{Ae^{at} - b}{a}}$$

Excellent!

6. Suppose a college has an endowment of \$100 million and earns interest at a continuous rate of 6%, but spends money from that endowment at a rate of \$5 million in the coming year, and then \$1 million more than that for each additional year that passes. This situation can be represented by the differential equation $\frac{dp}{dt} = 0.06p - (5+t)$. Use Euler's method with $\Delta t = 2$ to approximate, to the nearest tenth of a million dollars, the balance in that school's endowment fund after 6 years have gone by.

$$\frac{dp}{dt} = .06p - (5+t)$$

$$\frac{\Delta p}{\Delta t} = .06(100) - (5+0)$$

$$\Delta p = (6-5)\cdot(2) = 2$$

$$\frac{\Delta p}{\Delta t} = .06(102) - (5+2)$$

$$\Delta p = (6.12 - 7)\cdot(2) = -1.76$$

$$\frac{\Delta p}{\Delta t} = .06(100.24) - (5+4)$$

$$\Delta p = (6.0144 - 9)\cdot(2) = -5.9712$$

in millions

t	p	Δp
0	100	2
2	102	-1.76
4	100.24	-5.9712
6	94.2688	

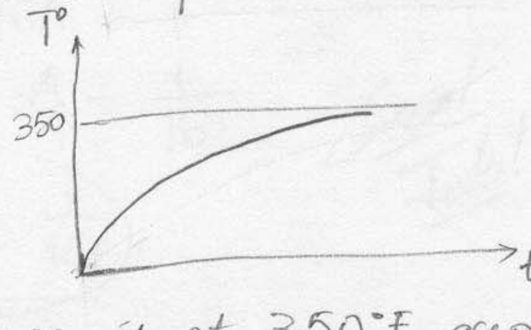
\$94.3 million

Excellent

7. Bunny is a calculus student at Anonymous State University, and she's having some trouble with differential equations. Bunny says "OHmygod, these are so confusing. We had this problem on our problem set where, like, we did the oily method, and then it asked about if there was an equilibrium, right? So we did the oily part, and it was totally obvious that it was going up slower and slower, right? So I said there wasn't an equilibrium, because it would never actually get there. The grader gave me, like, almost no points for that second part, which is totally wrong, because he said the first part was totally right. So if it doesn't get there, then it's not an equilibrium, right?

Explain clearly to Bunny whether her statements about equilibrium are correct, or if some refinement is in order.

Bunny was right in the first part and the result that she got was that "it was going up slower and slower" That result does not mean that there was no equilibrium. The result will get closer and closer to the equilibrium. Equilibrium occurs when rate of change equal zero. It means there are no change. For example:



The equilibrium is at 350°F according to this example. The temperature of the object is getting closer and closer to the equilibrium.

Hope that this would help you, Bunny!

It does! You clarified the distinction very nicely!

8. We used characteristic polynomials to solve differential equations of the form

$$ay'' + by' + cy = 0$$

but a variation of that approach can be used to solve equations like

$$y'' + 4y' - 5y = e^{3x}.$$

Find a solution to this equation by guessing that probably there's a solution of the form

$$y = Ae^{3x}$$

for some value of the coefficient A, and proceeding to find an appropriate value for A.

$y' = 3Ae^{3x}$

$y'' = 9Ae^{3x}$

$9Ae^{3x} + 4.3Ae^{3x} - 5Ae^{3x} = e^{3x}$

$9Ae^{3x} + 12Ae^{3x} - 5Ae^{3x} - e^{3x} = 0.$

$e^{3x}(9A + 12A - 5A - 1) = 0$

$16A - 1 = 0$ (Because e^{3x} cannot be equal to 0)

$16A = 1$

$A = \frac{1}{16}$

Great Job!

So: $A = \frac{1}{16}$

9. Water leaks out of a barrel at a rate proportional to the square root of the depth of the water at that time. If the water level starts out at 24 inches and drops to 20 inches in 1 hour, how long (to the nearest minute) will it take for half of the water to leak out of the barrel?

$$\frac{dw}{dt} = k \cdot \sqrt{w}$$ Separation of Variables!

$$\int \frac{dw}{\sqrt{w}} = \int k\, dt$$

$$2 \cdot w^{1/2} = kt + C$$

$$w^{1/2} = \frac{k}{2}t + D$$

General solution $w = \left(\frac{k}{2}t + D\right)^2$

To start with 24 inches:

$$24 = \left(\frac{k}{2} \cdot 0 + D\right)^2$$

$$\sqrt{24} = D$$

So $w = \left(\frac{k}{2}t + \sqrt{24}\right)^2$

To have 20 inches after an hour:

$$20 = \left(\frac{k}{2} \cdot 1 + \sqrt{24}\right)^2$$

$$\sqrt{20} = \frac{k}{2} + \sqrt{24}$$

$$k = 2 \cdot \sqrt{20} - 2\sqrt{24}$$

$$k \approx -.853687$$

So $w = (-.4268t + 4.8990)^2$

Then half the water is left when:

$$12 = (-.4268t + 4.8990)^2$$

$$3.4641 = -.4268t + 4.8990$$

$$t \approx 3.3616$$

So after about

3 hours, 22 minutes

10. Jon plans to spend his summer building elaborate fountain arrangements involving yard gnomes peeing into rain barrels. If the yard gnome trickles water into the barrel at a rate of 3 inches per hour,

a) What is the new differential equation, and what is its equilibrium depth (to the nearest tenth of an inch)?

b) Find a general solution to the new differential equation.

So $\frac{dw}{dt} = k\sqrt{w} + 3$ is the new equation,

$0 = k\sqrt{w} + 3$

$\frac{-3}{-.8537} = \sqrt{w}$ Constant for leak rate from problem 9

$w \approx 12.3494$

a) So the equilibrium depth is about 12.3 inches

And $\int \frac{1\,dw}{k\sqrt{w}+3} = \int dt$ let $u = k\sqrt{w} + 3 \implies u - 3 = k\sqrt{w}$

$\frac{du}{dw} = k \cdot \frac{1}{2} \cdot w^{-1/2}$ so $\sqrt{w} = \frac{u-3}{k}$

$\int \frac{1}{u} \cdot \frac{2w^{1/2}}{k} du = t + C$

$\frac{2w^{1/2}}{k} du = dw$

$\int \frac{1}{u} \cdot \frac{2}{k} \cdot \frac{u-3}{k} du = t + C$

$\frac{2}{k^2} \int \frac{u-3}{u} du = t + C$

$\int \left(1 - \frac{3}{u}\right) du = \frac{k^2}{2}(t + C)$

So $u - 3\ln|u| = \frac{k^2}{2}(t + C_2)$ is (an implicit) solution.

Where C_2 is a new constant incorporating the constant of integration from the other side.