Exam 1 Foundations of Advanced Mathematics 2/11/2005

Each problem is worth 10 points. Show appropriate justification for full credit. Don't panic.

1. State the definition of a odd integer.

An odd integer is any integer that can be represented as $2n+1$, where n is an integer.

Good

2. Let Λ be some indexing set. State the definition of $\bigcap_{\alpha\in\Lambda} B_\alpha$.

$$\bigcap_{\alpha\in\Lambda} B_\alpha = \{x : x \in B_\alpha \text{ for all } \alpha\in\Lambda\}$$

Excellent

3. Let A = {1, 2, 3, 4}, B = {1, 3, 5}, and C = {2, 3, 4}.

a) Compute $A \cap B$.

$A \cap B = \{1, 3\}$

b) Compute $\mathcal{P}$ (B), the power set of the set B.

$\mathcal{P}(B) = \{\{1\}, \{3\}, \{5\}, \{1,3\}, \{1,5\}, \{3,5\}, \{1,3,5\}, \emptyset\}$

Excellent

c) Compute $A \cup (B \setminus C)$

$B/C = \{1, 5\}$

$A \cup (B/C) = \{1, 2, 3, 4, 5\}$

4. Construct a truth table for the statement A ⇒ (B ∨ C).

A	B	C	B∨C	A⇒(B∨C)
T	T	T	T	T
T	F	T	T	T
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T
F	T	F	T	T
F	F	F	F	T

Great

5. Prove that the sum of an odd integer and an even integer is an odd integer.

Well, let there be an odd integer $2x+1$ where x is an integer and an even integer $2y$ where y is an integer, this is possible to define by the definitions of even and odd. Then the sum of an even and odd:

$$(2x+1)+2y$$

$$2x+2y+1$$

$$2(x+y)+1$$

let $x+y=z$ where z is an integer because the addition of integers yield integers

then $2(x+1)+2y = 2z+1$

$2z+1$ is an odd integer because it can be written in the form of 2 times an integer plus 1. So the sum of an odd and an even integer is an odd integer.

Nice Job!

6. Prove that the product of any three consecutive integers is divisible by 6.

Let there be three consecutive integers $n, n+1, n+2$ such that n is an integer. 6 is the product of the two prime numbers 2 and 3 so any product containing these two prime numbers will be divisible by 6. Consider $n(n+1)(n+2)$, the product of of the three consecutive integers $n, n+1, n+2$, n must be either even or odd so:

if n is odd it can be written as $n = 2x+1$ for any integer x then the next consecutive integer is $n+1$ so $n+1 = (2x+1)+1 = 2x+2 = 2(x+1)$ and let $x+1 = y$ where y is an integer since any integer plus 1 will yield another integer, then $n+1$ is even because it can be written as $n+1 = 2y$, def of even. If $n+1$ is even it is divisible by 2 ($n+1 = 2y$) so we found the prime factor 2.

Wow! Quite thorough!

if n is even, it can be written $n = 2x$ for any integer x and it is divisible by 2 ($n = 2x$) so again we found the prime factor 2

then for $n(n+1)(n+2)$ to be divisible by 6 it must also contain a factor of 3 so either $n, n+1$ or $n+2$ would have to be divisible by 3:

if n is divisible by 3 then we have our factor of 3

if n is not it is either $n = 3y+1$ (for any integer y) with a remainder of 1 or $n = 3y+2$ with a remainder of 2

if $n = 3y+1$ $n+1 = 3y+2$ and $n+2 = 3y+3 = 3(y+1)$ let $y+1 = z$ (for any integer z b/c an integer plus one yields another integer) then $n+2 = 3z$ which is divisible by 3

if $n = 3y+2$ then $n+1 = 3y+3 = 3(y+1) = 3z$ so $n+1$ is divisible by 3

Given all of the possibilities for $n, n+1$, and $n+2$ each possibility contains the factors (and primes) 2 and 3 so the product of any three consecutive integers is divisible by 6.

7. Biff is a student taking a math class at Anonymous State University, and he's having some trouble. Biff says "Dude, they just make this math stuff so confusing. So there's this thing in our math book, it says that there's this thing called the twin prime conjection. It says, like, that there's infinitely many times that there's prime numbers just two apart, like how both 11 and 13 are both of 'em prime. So I looked at it for a while, and figured out that it isn't true, 'cause like with 101 and 103, they're both prime. But with 1001 and 1003, you might think they're both prime, but they're not, 'cause 1003 has a 7 in it. So I don't know why the book makes it seem so complicated, when all you gotta do is try factoring a few numbers and you find some."

Explain clearly to Biff how what he says either does or does not refute or confirm the Twin Prime Conjecture (which claims, as Biff mentions, that there are infinitely many pairs of prime numbers with each pair consisting of two numbers which are two units apart from one another).

Well Biff your statement is correct but it does not refute the Twin Prime Conjecture. You must have gotten confused in the wording. Let me explain. It says that there are infinately many times that work. It does not say that every 2 odd numbers in a row will work. So just b/c 1001 and 1003 doesn't work there can still be infinately many consecutive prime numbers. If the statement were to say that every two prime numbers are consecutive then the statement would be false, but it does not. It just says there are infinately many. So try reading the problems more carefully and think what the theorem is actually saying. This will make math a little easier.

Great

8. Let A, B, and C be sets. Prove that $A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$.

$A \cup (B \cap C) \supseteq (A \cup B) \cap (A \cup C)$

$(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

So to prove that $(A \cup B) \cap (A \cup C)$ is a subset of $A \cup (B \cap C)$

let $x \in (A \cup B) \cap (A \cup C)$

then $x \in (A \cup B)$ and $x \in (A \cup C)$ because it is in the intersection of the two sets

then $x \in A$ or $x \in B$ because it is in the union of $A \cup B$
and $x \in A$ or $x \in C$ because it is in the union $A \cup C$

if $x \in A$ then $x \in A \cup (B \cap C)$ so $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ because all elements in $(A \cup B) \cap (A \cup C)$ are in $A \cup (B \cap C)$

if $x \in B$ also $x \in A$ or $x \in C$

if $x \in A$ the logic follows as above

if $x \in C$ then $x \in B \cap C$ because x is in both sets and so $x \in A \cup (B \cap C)$ and therefore $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ because all elements in $(A \cup B) \cap (A \cup C)$ are in $A \cup (B \cap C)$

Excellent

Since all possibilities of elements x in $(A \cup B) \cap (A \cup C)$ are all in $A \cup (B \cap C)$, $(A \cup B) \cap (A \cup C)$ must be a subset of $A \cup (B \cap C)$ by definition of subset.

9. Let A, B, and C be sets. Prove or give a counterexample to the statement:

$$\text{If } A \subseteq B, \text{ then } A \cap C \subseteq B \cap C.$$

let $x \in (A \cap C)$

then $x \in A$ or $x \in C$

if $x \in C$ then $x \in B \cap C$ so the element x of $A \cap C$ will be in $B \cap C$

if $x \in A$, $x \in B$ because $A \subseteq B$ and by definition of subsets all elements in A are also in B

Since $x \in A$ and $x \in B$ $x \in A \cap C$ and $x \in B \cap C$

So for all elements $x \in A \cap C$, they are all also in $B \cap C$, since all elements in $A \cap C$ are in $B \cap C$

so

$A \cap C \subseteq B \cap C$ by definition of subset.

Beautiful.

10. Let A be a set of real numbers. Define an element g of A to be the **greatest element** in A if for any $a \in A$, $g \geq a$. Prove that the greatest element in A is unique.

Well, s'pose the greatest element in A isn't unique, so there are two greatest elements g_1 and g_2. Then since g_1 is a greatest element in A, we know $g_1 \geq a$ for all $a \in A$, and in particular because $g_2 \in A$ we have $g_1 \geq g_2$. But similarly since g_2 is a greatest element in A and $g_1 \in A$ we have $g_2 \geq g_1$. And the only way we can have $g_1 \geq g_2$ and $g_2 \geq g_1$ is if $g_1 = g_2$, so in fact that greatest element is unique. $\square$