Exam 2 Foundations of Advanced Mathematics 3/25/2005

Each problem is worth 10 points. Show appropriate justification for full credit. Don’t panic.

1. State the definition of a reflexive relation ~ on a set A.

2. State the definition of a one-to-one function $f:A \to B$.

3. Give an example of a function $f:\mathbb{R} \to \mathbb{R}$ which is one-to-one but not onto.

4. Let $f:\mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

a) Find $f([1,3])$.

b) Find $f^{-1}((1,4])$.

5. Give examples of:

a) A graph and a walk in the graph that is not a path.

b) Two different trees with 7 vertices.

6. Consider the relation $(a, b) \sim (c, d)$ iff $ad = bc$. Is this relation:

a) reflexive?

b) symmetric?

c) transitive?

d) an equivalence relation on $\mathbb{R}$?

7. Bunny is a student taking a math class at Anonymous State University, and she's having some trouble. Bunny has written the following proof that $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$:

"Note that if $n = 1$ the statement reads $1^3 = \left[\frac{1(1+1)}{2}\right]^2$ which is true. So suppose the statement is true for some natural number k. Then $k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 =$

$$\frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} = \frac{(k+1)^2(k^2+4(k+1))}{4} = \frac{(k+1)^2(k^2+4k+4)}{4} =$$

$$\frac{(k+1)^2(k+2)^2}{4} = \left[\frac{(k+1)((k+1)+1)}{2}\right]^2.$$ Therefore by the principle of mathematical induction the statement is true for all natural numbers." [from *Primus*, September 2004, p. 256]

Critique Bunny's proof. Your job is not to provide a proof yourself, but rather to **explain as clearly as possible to Bunny** what is satisfactory or unsatisfactory about her proof.

8. Prove that for all $n \in \mathbb{N}$, 3 divides $4^n - 1$.

9. Prove that if $f:A \to B$ is a function and T_1 and T_2 are subsets of A, then

$$f(T_1 \cap T_2) \subseteq f(T_1) \cap f(T_2).$$

10. Prove or give a counterexample to the following proposition: If f, g, and h are functions such that $f:B \to C$, $g:A \to B$, and $h:A \to B$, with $f \circ g = f \circ h$, then $g = h$.

Extra Credit (5 points possible): Let K_n be the **complete graph on *n* vertices**, that is, a graph with *n* vertices and with edges connecting every pair of vertices. How many edges does K_n have?