Each problem is worth 10 points. Show appropriate justification for full credit. Don’t panic.

1. State the definition of a reflexive relation ~ on a set A.

   \[ n \text{ is said to be reflexive if for all } x \in A, x \sim x. \]

2. State the definition of a one-to-one function \( f: A \to B \).

   A function \( f: A \to B \) is one-to-one if for all \( b \in B \), there is at most one \( a \in A \) such that \( f(a) = b \).

3. Give an example of a function \( f: \mathbb{R} \to \mathbb{R} \) which is one-to-one but not onto.

   \[ f(x) = e^x \text{ for } x \in \mathbb{R}. \]

   Is one-to-one but doesn’t hit the negative values of \( \mathbb{R} \).

   Excellent!
4. Let \( f : \mathbb{R} \to \mathbb{R} \) be given by \( f(x) = x^2 \).
   
a) Find \( f([1,3]) \).
   
   \[
   f([1,3]) = [1, 9]
   \]

   b) Find \( f^{-1}((1,4]) \).
   
   \[
   f^{-1}((1,4]) = [-2, -1) \text{ and } (1, 2]
   \]

5. Give examples of:
   
a) A graph and a walk in the graph that is not a path.
   
   \[
   (A, B), (B, C), (C, D), (D, B), (B, E)
   \]

   b) Two different trees with 7 vertices.

6. Consider the relation \((a, b) \sim (c, d)\) iff \(ad = bc\). Is this relation:

a) reflexive? Yes

We need to know if \((a, b) \sim (a, b)\), which means \(a \cdot b = b \cdot a\), which is true.

b) symmetric? Yes

We need to know if whenever \((a, b) \sim (c, d)\) it also happens that \((c, d) \sim (a, b)\). But \((a, b) \sim (c, d)\) means \(a \cdot d = b \cdot c\) and \((c, d) \sim (a, b)\) means \(c \cdot b = d \cdot a\), and those are the same.

c) transitive? Yes

We need to know if whenever \((a, b) \sim (c, d)\) and \((c, d) \sim (e, f)\), it also happens that \((a, b) \sim (e, f)\). But those mean \(a \cdot d = b \cdot c\) and \(c \cdot f = d \cdot e\), or rearranging \(\frac{a}{b} = \frac{d}{c}\) and \(\frac{c}{d} = \frac{e}{f}\), and applying the transitive property of real numbers to these equations produces \(\frac{a}{b} = \frac{e}{f}\) or \(a \cdot f = b \cdot e\), which means \((a, b) \sim (e, f)\).

d) an equivalence relation on \(\mathbb{R}\)?

Yes*, since it's reflexive, symmetric, and transitive.

* The hitch is if \(b\) or \(d\) is zero, in which case dividing both sides by them isn't possible. So it's not actually transitive if we use points like \((1,0)\), \((0,0)\), and \((0,2)\), but with the requirement that \(b, d,\) and \(f\) to it's transitive.
7. Bunny is a student taking a math class at Anonymous State University, and she’s having some trouble. Bunny has written the following proof that 

\[ 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2 \]

“Note that if \( n = 1 \) the statement reads \( 1^3 = \left[ \frac{1(1+1)}{2} \right]^2 \) which is true. So suppose the statement is true for some natural number \( k \). Then 

\[ k^3 + (k+1)^3 = \left[ \frac{k(k+1)}{2} \right]^2 + (k+1)^3 = \]

\[ \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} = \frac{(k+1)^2(k^2 + 4k + 4)}{4} = \frac{(k+1)^2(k+2)^2}{4} = \left[ \frac{(k+1)((k+1)+1)}{2} \right]^2 \]. Therefore by the principle of mathematical induction the statement is true for all natural numbers.” [from Primus, September 2004, p. 256]

Critique Bunny’s proof. Your job is not to provide a proof yourself, but rather to explain as clearly as possible to Bunny what is satisfactory or unsatisfactory about her proof.

Well Bunny, what you did proved is not exactly what you stated. What you need to prove is that 

\[ \sum_{i=1}^{n} i^3 = \left[ \frac{n(n+1)}{2} \right]^2 \]. What needs to be proved is that 

\[ \sum_{i=1}^{k+1} i^3 = \left[ \frac{(k+1)(k+2)}{2} \right]^2 \]. When you proved it for \( n \), you did not include the numbers that preceded it in the series \( k \) in the series.
8. Prove that for all \( n \in \mathbb{N} \), 3 divides \( 4^n - 1 \).

\[ \text{n = some integer} \]

**Induction**

First show for \( n = 1 \), \( 4^1 - 1 = 3 \), 3 is divisible by 3. So we assume this is true for \( n \) values, hence our inductive hypothesis is \( 4^n - 1 \). Let's now show for \( n+1 \) values.

\[ 4^{n+1} - 1 = 3p, \ p \text{ being some integer} \]

\[ 4 \cdot 4^n - 1 = 3p \]

\[ 4^n + 4^n + 4^n + (4^n - 1) = 3p \]

well, from our inductive hyp. we know that \( 4^n - 1 \) is divisible by 3, so we must find that \( 4^n + 4^n + 4^n \) is divisible by 3. We can rewrite that as \( 3(4^n) \). Since we know that any integer, in this case 4, taken to the power of some integer creates an integer, we can rewrite that statement as \( 3q \), where \( q \) is some integer. Any integer multiplied by 3 is divisible by 3 so the \( n+1 \) values work, proving that for all \( n \in \mathbb{N} \), 3 divides \( 4^n - 1 \).

*Well done!*
9. Prove that if \( f: A \to B \) is a function and \( T_1 \) and \( T_2 \) are subsets of \( A \), then

\[
f(T_1 \cap T_2) \subseteq f(T_1) \cap f(T_2).
\]

We'll argue elementwise. Take \( b \in f(T_1 \cap T_2) \). Then \( b = f(t) \) for some \( t \in T_1 \cap T_2 \), which means \( t \in T_1 \) and \( t \in T_2 \). Then we also have \( b \in f(T_1) \) and \( b \in f(T_2) \), so \( b \in f(T_1) \cap f(T_2) \). Since any element of \( f(T_1 \cap T_2) \) is also in \( f(T_1) \cap f(T_2) \), we have \( f(T_1 \cap T_2) \subseteq f(T_1) \cap f(T_2) \) as desired. \( \square \)

10. Prove or give a counterexample to the following proposition: If \( f, g, \) and \( h \) are functions such that \( f: B \to C \), \( g: A \to B \), and \( h: A \to B \), with \( f \circ g = f \circ h \), then \( g = h \).

Consider the following counterexample: