Exam 2 Foundations of Advanced Mathematics 3/25/2005

Each problem is worth 10 points. Show appropriate justification for full credit. Don't panic.

1. State the definition of a reflexive relation ~ on a set A.

$\sim$ is said to be reflexive if for all $x \in A$, $x \sim x$.

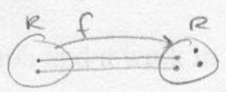
Good

2. State the definition of a one-to-one function f:A → B.

A function $f: A \to B$ is one-to-one if for all $b \in B$, there is at most one $a \in A$ such that $f(a) = b$

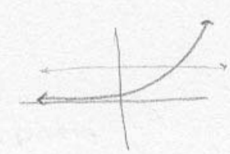
Great

3. Give an example of a function f:ℝ → ℝ which is one-to-one but not onto.



$f: \mathbb{R} \to \mathbb{R}$

$f(x) = e^x$ for $x \in \mathbb{R}$

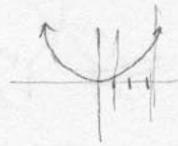


Is one-to-one but doesn't hit the negative values of $\mathbb{R}$.

Excellent!

4. Let $f:\mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.

a) Find $f([1,3])$.

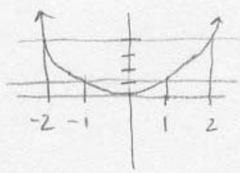


$[1, 9]$

$f([1, 3]) = [1, 9]$

Great

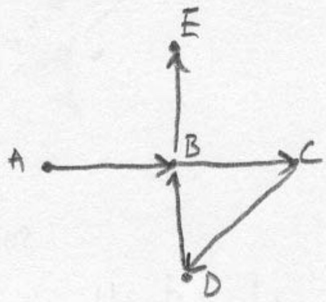
b) Find $f^{-1}((1,4])$.



$f^{-1}((1,4]) = [-2, -1)$ and $(1, 2]$

5. Give examples of:

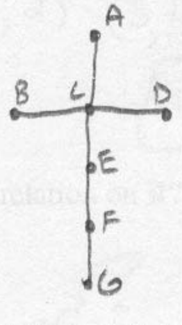
a) A graph and a walk in the graph that is not a path.



(A, B), (B, C), (C, D), (D, B), (B, E)

Yes!

b) Two different trees with 7 vertices.



Good

6. Consider the relation $(a, b) \sim (c, d)$ iff $ad = bc$. Is this relation:

a) reflexive?

Yes

We need to know if $(a,b) \sim (a,b)$, which means $a \cdot b = b \cdot a$, which is true.

b) symmetric?

Yes

We need to know if whenever $(a,b) \sim (c,d)$ it also happens that $(c,d) \sim (a,b)$. But $(a,b) \sim (c,d)$ means $a \cdot d = b \cdot c$ and $(c,d) \sim (a,b)$ means $c \cdot b = d \cdot a$, and those are the same.

c) transitive?

Yes

We need to know if whenever $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, it also happens that $(a,b) \sim (e,f)$. But those mean $a \cdot d = b \cdot c$ and $c \cdot f = d \cdot e$, or rearranging* $\frac{a}{b} = \frac{c}{d}$ and $\frac{c}{d} = \frac{e}{f}$, and applying the transitive property of real numbers to these equations produces $\frac{a}{b} = \frac{e}{f}$ or $a \cdot f = b \cdot e$, which means $(a,b) \sim (e,f)$.

d) an equivalence relation on $\mathbb{R}$?

Yes,* since it's reflexive, symmetric, and transitive.

* The hitch is if b or d or f is zero, in which case dividing both sides by them isn't possible. So it's not actually transitive if we use points like $(1,0)$, $(0,0)$, and $(0,2)$, but with the requirement that b, d, and $f \neq 0$ it's transitive.

7. Bunny is a student taking a math class at Anonymous State University, and she's having some trouble. Bunny has written the following proof that $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2}\right]^2$:

"Note that if $n = 1$ the statement reads $1^3 = \left[\frac{1(1+1)}{2}\right]^2$ which is true. So suppose the statement is true for some natural number k. Then $k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 =$

$$\frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} = \frac{(k+1)^2(k^2+4(k+1))}{4} = \frac{(k+1)^2(k^2+4k+4)}{4} =$$

$\frac{(k+1)^2(k+2)^2}{4} = \left[\frac{(k+1)((k+1)+1)}{2}\right]^2$. Therefore by the principle of mathematical induction the statement is true for all natural numbers." [from *Primus*, September 2004, p. 256]

Critique Bunny's proof. Your job is not to provide a proof yourself, but rather to **explain as clearly as possible to Bunny** what is satisfactory or unsatisfactory about her proof.

Well Bunny, what you ~~did~~ proved is not exactly what you stated.

What you need to prove is that $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$. Yes! What needs to be proved is that $\sum_{i=1}^{k} i^3 + (k+1)^3 = \sum_{i=1}^{k+1} i^3 = \left[\frac{(k+1)(k+1+1)}{2}\right]^2$.

When you proved it for $(k+1)$, you did not include the numbers that ~~precede~~ could proceed ~~in the series~~ k in the series.

Great!

D

8. Prove that for all $n \in \mathbb{N}$, 3 divides $4^n - 1$. n = some integer

Induction

First show for $n=1$. $4^1-1=3$. 3 is divisible by 3. So we assume this is true for n values, hence our inductive hypothesis is 4^n-1. Lets now show for n+1 values.

$4^{n+1}-1 \stackrel{?}{=} 3p$, p being some integer.

$4 \cdot 4^n - 1 = 3p$

$4^n+4^n+4^n+(4^n-1)=3p$

well, from our inductive hyp. we know that 4^n-1 is divisible by 3. So we must find that $4^n+4^n+4^n$ is divisible by 3. We can rewrite that as $3(4^n)$. Since we know that any integer, in this case 4, taken to the power of some integer creates an integer, we can rewrite that statement as 3Q where Q is some integer. Any integer multiplied by 3 is divisible by 3 so the n+1 values work, proving that for all $n \in \mathbb{N}$, 3 divides 4^n-1.

Well done!

9. Prove that if f:A → B is a function and T_1 and T_2 are subsets of A, then

$$f(T_1 \cap T_2) \subseteq f(T_1) \cap f(T_2).$$

We'll argue elementwise. Take $b \in f(T_1 \cap T_2)$. Then $b = f(t)$ for some $t \in T_1 \cap T_2$, which means $t \in T_1$ and $t \in T_2$. Then we also have $b \in f(T_1)$ and $b \in f(T_2)$, so $b \in f(T_1) \cap f(T_2)$. So since any element of $f(T_1 \cap T_2)$ is also in $f(T_1) \cap f(T_2)$, we have $f(T_1 \cap T_2) \subseteq f(T_1) \cap f(T_2)$ as desired. □

10. Prove or give a counterexample to the following proposition: If f, g, and h are functions such that f:B → C, g:A → B, and h:A →B, with f ∘ g = f ∘ h, then g = h.

Consider the following counterexample: