Exam 2 Calc 1 3/2/2006

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. State and prove the Sum Rule for derivatives.

if both functions $f(x)$ and $g(x)$ are differentiable, then $(f+g)'(x) = f'(x) + g'(x)$

Proof: well,

$$(f+g)'(x) = \lim_{h \to 0} \frac{(f+g)(x+h) - (f+g)(h)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(h) - g(h)}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{f(x+h) - f(h)}{h}} + \underbrace{\frac{g(x+h) - g(x)}{h}}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(h)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x) \quad \square$$

Excellent!

2. Find an equation for the line tangent to the function $f(x) = x^3 - \ln x$ at the point where $x = 1$.

$f'(x) = 3x^2 - 1/x$

$f'(1) = 3(1)^2 - 1/(1)$ where $x = 1$

$f'(1) = 3 - 1 = \underline{2} = m$ (slope)

If $x = 1$, then $\underline{f(1) = 1 - \ln 1 = 1}$.

So the equation of the line passing $(1,1)$ with slope 2 is

$(y-1) = 2(x-1)$

$\Rightarrow y - 1 = 2x - 2$

$\boxed{y = 2x - 1}$

Great

$\frac{1}{\sinh x}$ $\coth = \frac{\cosh}{\sinh}$

3. Show that the derivative of csch x is –csch x coth x.

$$(\operatorname{csch} x)' = -\operatorname{csch} x \coth x$$

Quotient Rule

$$\frac{f' \cdot g - f \cdot g'}{g^2}$$

$$\left(\frac{1}{\sinh x}\right)' = \frac{0(\sinh x) - (1)(\cosh x)}{(\sinh x)^2}$$

$$= \frac{0 - \cosh x}{(\sinh x)^2}$$

$$= \frac{-\cosh x}{(\sinh x)^2}$$

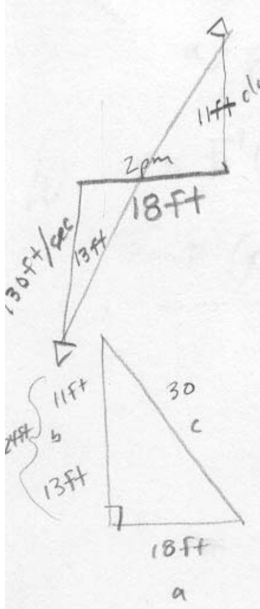
$$= -\frac{\cosh x}{(\sinh x)(\sinh x)}$$

$$= -\frac{1}{\sinh x} \cdot \frac{\cosh x}{\sinh x} = \boxed{-\operatorname{csch} x \cdot \coth x}$$

Excellent!

4. A busload full of clowns in the northbound lane of a divided highway passes a busload full of mimes in the southbound lane at exactly 2pm. If the clowns are traveling at 110 feet per second and the mimes are traveling 130 feet per second, and the northbound and southbound lanes are parallel and 18 feet apart, at what rate is the distance between the clowns and the mimes changing a tenth of a second later, at the sublime moment when one of the mimes glimpses one of the clowns through the back window and falls forever in love?

W



Nice Job!

$$a^2 + b^2 = c^2$$

$$18^2 + 24^2 = c^2$$

$$324 + 576 = c^2$$

$$900 = c^2$$

$$c = 30$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

$$2 \cdot 18 \cdot 0 + 2 \cdot 24 \cdot 240 = 2 \cdot 30 \cdot \frac{dc}{dt}$$

$$\frac{11520}{60} = \frac{60}{60}\frac{dc}{dt}$$

$$\boxed{\frac{dc}{dt} = 192 \text{ ft/sec}}$$

$$\frac{da}{dt} = 0 \qquad \frac{db}{dt} = 130 + 110 = 240 \qquad \frac{dc}{dt} = ?$$

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$$

$$\frac{f' \cdot g - f \cdot g'}{g^2}$$

$$\frac{\cos x \cdot \cos x - \sin x \cdot -\sin x}{\cos^2 x}$$

$$\frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \sec^2 x$$

5. Show that $\left(\tan^{-1} x\right)' = \dfrac{1}{1+x^2}$.

Implicit differentiate

$$\boxed{\tan(\tan^{-1} x)}' = \boxed{x}'$$

$$\sec^2(\tan^{-1} x) \cdot (\tan^{-1} x)' = 1$$

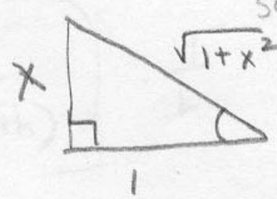
$$(\tan^{-1} x)' = \frac{1}{\sec^2(\tan^{-1} x)}$$

This means an angle such that the tangent is x.

Then what is the $\sec^2$ of that triangle.

SOH CAH TOA

Excellent



so $\dfrac{1}{\sec^2(\tan^{-1} x)} = \dfrac{1}{1+x^2}$

$a^2 + b^2 = c^2$

cos is $\left(\dfrac{1}{\sqrt{1+x^2}}\right)^2 = \dfrac{1}{1+x^2}$ so $\sec^2 = 1+x^2$

6. a) Suppose $f(x)$ is a differentiable function and $F(x) = \sqrt{f(x)}$. Express $F'(x)$ in terms of $f(x)$ and $f'(x)$.

b) Suppose $g(x)$ is a differentiable function and $G(x) = \ln(g(x))$. Express $G'(x)$ in terms of $g(x)$ and $g'(x)$.

(a). Chain Rule

$F(x) = (f(x))^{1/2}$

$$\boxed{F'(x) = \tfrac{1}{2}(f(x))^{-1/2} \cdot f'(x)}$$

(b). $G(x) = \ln(g(x))$ Chain Rule

$G'(x) = \left(\dfrac{1}{g(x)}\right) \cdot g'(x)$

Great!

$$\boxed{G'(x) = \frac{g'(x)}{g(x)}}$$

7. State and prove the Quotient Rule for derivatives.

If $f(u)$ & $g(u)$ are differentiable and $g(u) \neq 0$

then $\left(\frac{f}{g}\right)'(u) = \frac{f'g - fg'}{g^2}$

Proof: well,

consider $h(u) = \frac{1}{g(u)}$

$$h'(u) = \lim_{h \to 0} \frac{\frac{1}{g(u+h)} - \frac{1}{g(u)}}{h}$$

$$h'(u) = \lim_{h \to 0} \frac{g(u) - g(u+h)}{h \cdot g(u) g(u+h)}$$

$$h'(u) = \lim_{h \to 0} \frac{-(g(u+h) - g(u))}{h} \cdot \lim_{h \to 0} \frac{1}{g(u) \cdot g(u+h)}$$

$$h'(u) = -g'(u) \cdot \frac{1}{g^2(u)}$$

$$h'(u) = \frac{-g'(u)}{g^2 u}$$

now consider $\left(f \cdot \frac{1}{g}\right)' u$

$$= (f)'(u) \cdot \frac{1}{g(u)} + f(u) \cdot \left(\frac{1}{g(u)}\right)'$$

$$= f'(u) \frac{1}{g(u)} + f(u) \cdot \frac{-g'(u)}{g^2(u)}$$

Great

$$= \frac{f'(u) g(u) - f(u) g'(u)}{g^2(u)}$$

$$= \frac{f'g - fg'}{g^2} . \square$$

8. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dang, this calculus stuff is kicking my ass. I liked it when it was just formulas and stuff, but now there's all this stuff where it's not just working it out. The professor keeps talking about reasoning, like that's a good thing or something. So I guess he was trying to convince us or something, and he said this thing about if you know the derivative of a function, then you also know the derivative of its inverse. It sounded like he thought it was important, but I couldn't figure out what the heck he meant."

Not Nice

Explain clearly to Biff how knowing the derivative of $f(x)$ tells us something about the derivative of $f^{-1}(x)$.

Biff - by knowing the derivative of a $f(x)$, we can use that information to help us find the derivative of $f^{-1}(x)$. We do this by setting up an equation that we know is true such as $\sin(\sin^{-1} x) = x$ we can then implicitly differentiate this equation so that we can figure out what the derivative of $\sin^{-1} x$ is.

So using the Chain Rule we take the derivative of sin of the inside function, times the derivative of the inside function. And we know that will equal the derivative of x, which is 1.

$$\cos(\sin^{-1} x) \cdot (\sin^{-1} x)' = 1$$

We then solve this equation for $(\sin^{-1} x)'$ and if we want we can take it to a picture of the triangle to make the answer simplier. Yes!

9. Find a second-degree polynomial (so a function of the form $f(x) = ax^2 + bx + c$) for which $f(0) = 5, f'(0) = -1$, and $f''(0) = 2$.

$F(x) = ax^2 + bx + c$

$F(0) = c$

$F'(x) = 2ax + b$

$f'(0) = b$

$f''(x) = 2a$

$F''(0) = 2a$

we know $f(0) = 5$

So, $\underline{c = 5}$

$F'(0) = -1$

so $\underline{b = -1}$

$f''(0) = 2$

$2a = 2$

$\underline{a = 1}$

Excellent

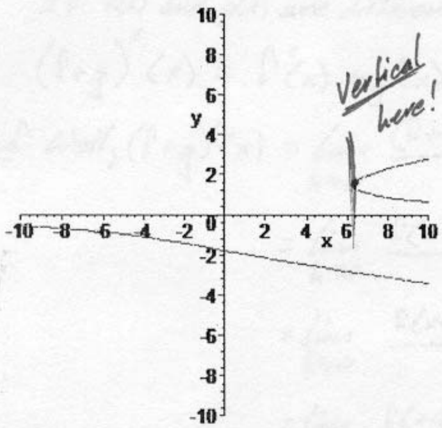
So the 2^{nd} degree polynomial would be

$\underline{x^2 - x + 5}$

10. The graph of the curve with equation $y^3 - xy = -6$ is shown below.

a) Find the slope of the line tangent to this curve.

b) The graph has an "upper loop". Find the exact coordinates of the leftmost point on it.



$$3y^2 \cdot y' - (1 \cdot y + x \cdot y') = -0$$

$$3y^2 \cdot y' - xy' = y$$

$$y'(3y^2 - x) = y$$

a) $$y' = \frac{y}{3y^2 - x}$$

b) Well, the slope of the tangent line will be undefined there, so it must be a place where y' has a denominator of zero.

$$0 = 3y^2 - x$$

$$x = 3y^2$$

But it's also a point on our curve, so I'll substitute that into the curve's equation:

$$y^3 - (3y^2)y = -6$$

$$y^3 - 3y^3 = -6$$

$$-2y^3 = -6$$

$$y^3 = 3$$

$$y = \sqrt[3]{3}$$

And to find x: $x = 3(\sqrt[3]{3})^2$

So $(3\sqrt[3]{9}, \sqrt[3]{3})$