*C → mental note to myself to not forget the C in antiderivatives!

**Exam 3** **Calc 1** **3/31/2006**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. If $f(x) = 9x - x^2$, what is the maximum value of $f(x)$?

$f'(x) = 9 - 2x$ → Take derivative, then set equal to zero.

$0 = 9 - 2x$

$x = 9/2$

↳ Plug into original equation!

Excellent!

$f(9/2) = 9(9/2) - (9/2)^2 = \boxed{81/4}$

2. If $g'(x) = 2x - e^x + \sin x$, find the most general antiderivative of $g'(x)$.

$G(x) = x^2 - e^x - \cos x + C$ Good

check: $G'(x) = 2x - e^x + \sin x$

3. If a company's cost to produce $x$ widgets is given by $C(x) = 5000 + 200x + 2x^2$, what are
   a. The company's average cost at a production level of 100 widgets?

Avg cost $= \frac{C(x)}{x} = \frac{5000 + 200x + 2x^2}{x}$

$\frac{C(100)}{100} = \frac{5000 + 200(100) + 2(100)^2}{100}$

$\boxed{= \$450}$

   b. The company's marginal cost at a production level of 100 widgets?

Marginal cost $= C'(x) = 200 + 4x$

Great

$C'(100) = 200 + 4(100)$

$= \$600$

4. Find the intervals of concavity and inflection point(s) of $f(x) = x e^{-x}$.

$f(x) = xe^{-x}$

$f'(x) = (1)e^{-x} + x(-e^{-x})$

$= e^{-x}(1-x)$

$f''(x) = -e^{-x}(1-x) + e^{-x}(-1)$

$= e^{-x}(-1+x-1)$

$= e^{-x}(x-2)$ → set equal to zero

$0 = e^{-x}(x-2)$

$x = 2$

| Interval | $f''(x)$ | concavity |
|---|---|---|
| $x<0$ | $-$ | downward |
| $0<x<2$ | $-$ | downward |
| $x>2$ | $+$ | upward |

CU for $(2,\infty)$
CD for $(-\infty,2)$

Inflection Point at $x=2$

$f(2) = \frac{2}{e^2}$, so $\left(2, \frac{2}{e^2}\right)$ = IP

Excellent!

5. Evaluate $\lim_{x\to 0^+} x \ln x$

$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} \overset{\text{use L'Hôpital}}{=} \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} \frac{-x^2}{x}$

$\lim_{x\to 0^+} \frac{-x}{1} = \frac{0}{1} = 0$

Nice Job!

6. Find the absolute maximum and minimum values of $f(t) = t\sqrt{5 - t^2}$ on the interval ~~[0,4]~~ [0,2].

Take the derivative and set it equal to zero.

$f'(t) = 1(5-t^2)^{1/2} + t\frac{1}{2}(5-t^2)^{-1/2} \cdot -2t$

$= (5-t^2)^{1/2} + t^2(5-t^2)^{-1/2}$

$0 = \sqrt{5-t^2} - \frac{t^2}{\sqrt{5-t^2}}$

$0 = 5 - t^2 - t^2$

$-5 = -2t^2$

$\frac{5}{2} = t^2, \quad t = \pm\sqrt{5/2}$, disregard $-\sqrt{5/2}$ b/c it is not on the interval.

$f(0) = 0\sqrt{5-0^2} = 0$ → absolute minimum

vs. $f(\sqrt{5/2}) = \sqrt{5/2}\sqrt{5-(\sqrt{5/2})^2} = 5/2$ → absolute maximum

vs. $f(2) = 2\sqrt{5-(2)^2} = 2$

Wonderful!

7. Use Newton's Method to approximate $\sqrt[3]{5}$. Start with $x_1 = 2$, and compute $x_2$ and $x_3$ to the nearest thousandth.

$\sqrt[3]{5}$ $\quad x^3 = 5 \quad f(x) = x^3 - 5 \quad f'(x) = 3x^2$

$x_2 = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{3}{12} = \frac{7}{4}$ or 1.75

$x_3 = \frac{7}{4} - \frac{f(\frac{7}{4})}{f'(\frac{7}{4})} = \frac{7}{4} - \frac{23/64}{147/16} = \frac{503}{294}$ or 1.711

Excellent

8. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this calculus stuff is so weird. We had this question on our test yesterday, and it was like, the answer was given, but you had to say *why*. I mean, I always thought it was the answer if the book said it was, right? So it was like, why is arcsin ($x/a$) the antiderivative for $\dfrac{1}{\sqrt{a^2-x^2}}$? Which totally confuses me, because that was on the homework and someone asked about it in our discussion section, and the TA just said that's what it was. So isn't it totally unfair that we're supposed to know *why*?

Explain clearly to Bunny how you know the antiderivative of $\dfrac{1}{\sqrt{a^2-x^2}}$ is arcsin ($x/a$).

Well, to say $F(x)$ is an antiderivative of $f(x)$ just means that the derivative of $F(x)$ is $f(x)$. So this question about antiderivatives really amounts to a question about derivatives: Does

$$\left[\arcsin\left(\frac{x}{a}\right)\right]' = \frac{1}{\sqrt{a^2-x^2}} \ ?$$

So we should work out $\left[\arcsin\left(\frac{x}{a}\right)\right]'$, which will use the Chain Rule:

$$\left[\arcsin\left(\frac{x}{a}\right)\right]' = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \left(\frac{x}{a}\right)'$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}$$

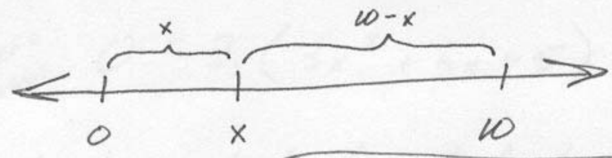
$$= \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a}$$

$$= \frac{1}{\frac{\sqrt{a^2-x^2}}{a}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{a^2-x^2}}$$

So since $\arcsin\left(\frac{x}{a}\right)$ has $\frac{1}{\sqrt{a^2-x^2}}$ for its derivative, we can also say that $\frac{1}{\sqrt{a^2-x^2}}$ has $\arcsin\left(\frac{x}{a}\right)$ for its antiderivative. □

9. The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 ft apart, **write an equation** whose solution will give the location where an object could be placed on the line between the sources so as to receive the least illumination. [Stewart 5th, p. 338].




Inversely proportional to square of distance from left endpoint

Inversely proportional to square of distance from right endpoint

$f(x) = k \cdot \frac{1}{x^2} + k \cdot \frac{3}{(10-x)^2}$

Take deriv: $f'(x) = -2k \cdot x^{-3} - 2k \cdot 3(10-x)^{-3}$

Set = zero: $0 = -2k\left(\frac{1}{x^3} + \frac{3}{(10-x)^3}\right)$

So this is an equation whose solution gives the location (in feet from the weaker source) of least illumination.

10. For what values of the constant $b$ does the function $f(x) = 2x^3 + bx^2 + 10x - 7$ have both a local maximum and local minimum point?

$f(x) = 2x^3 + bx^2 + 10x - 7$

$f'(x) = 6x^2 + 2bx + 10$

$0 = 6x^2 + 2bx + 10$

$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\frac{-(2b) \pm \sqrt{(2b)^2 - 4(6)(10)}}{2(6)}$

$\frac{-2b \pm \sqrt{4b^2 - 240}}{12}$

$4b^2 - 240 = 0$

$4b^2 = 240$

$b^2 = 60$

$b = \pm\sqrt{60}$

There will be a local max & min for any number $x > \sqrt{60}$ and $x < -\sqrt{60}$.

Excellent.