1. a) State the definition of an odd integer.

An odd integer \( n \) can be written in the form \( n = 2m + 1 \) for some integer \( m \).

b) Suppose that \( n \) is an integer. Show that if \( n^2 \) is even, then \( n \) is even.

\[ n^2 = 2a \text{ for some integer } a \]

\( n \) can either be even or odd.

- If \( n \) is even, it is in the form \( n = 2b \) for some integer \( b \)
  \[ n^2 = (2b)^2 = 4b^2 = 2(2b^2) \rightarrow 2b^2 = c \]
  \( c = 2c \) shows that an even squared is even

- If \( n \) is odd, it is in the form \( n = 2d + 1 \) for some integer \( d \).
  \[ n^2 = (2d+1)^2 = 4d^2 + 4d + 1 = 2(2d^2 + 2d) + 1 \rightarrow 2d^2 + 2d = e \]
  \( e = 2e + 1 \) shows that an odd squared is odd.

- Since these are the only two cases, the only way for \( n^2 \) to be even is if \( n \) is even.
2. a) Make a truth table for the statement \((P \land \neg Q) \rightarrow R\).

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>\neg Q</th>
<th>(P \land \neg Q)</th>
<th>(P \land \neg Q) \Rightarrow R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</table>


b) The propositional \(\neg (P \iff Q)\) is equivalent to \(\neg P \iff \neg Q\).

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P \iff Q</th>
<th>\neg (P \iff Q)</th>
<th>\neg P</th>
<th>\neg Q</th>
<th>(\neg P \iff \neg Q)</th>
</tr>
</thead>
<tbody>
<tr>
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These are not equal. You can't distribute a \(\neg\) sign. 
\(\neg (P \iff Q)\) would be equal to \((P \iff \neg Q)\) but that's not what it's asking.
3. $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is rational and therefore can be written in the form $\frac{a}{b}$, $a, b \in \mathbb{Z}$ and $a \neq b$ don't have any common factors.

So: $\sqrt{2} = \frac{a}{b}$

Square both sides: $2 = \frac{a^2}{b^2}$

rearrange to: $2b^2 = a^2$

Thus $a^2$ must be even making $a$ also even because an $Odd^2 = Odd$. Then $a$ can be written in the form $a = 2r$, $r \in \mathbb{Z}$

replacing the $a$ with $2r$: $2b^2 = 4r^2$

divide both sides by 2: $b^2 = 2r^2$

Hence $b^2$ must also be even forcing $b$ to be even which contradicts the fact that $a \neq b$ don't have common factors.

Thus proving $\sqrt{2}$ is irrational.

Wonderful
4. If $c$ is divisible by $b$, and $b$ is divisible by $a$, then $c$ is divisible by $a$.

\[\text{if } c \text{ is divisible by } b \text{ then } c = bn \text{ for } n \in \mathbb{Z}\]

\[\text{if } b \text{ is divisible by } a \text{ then } b = am \text{ for } m \in \mathbb{Z}\]

Prove $c = ap$ for $p \in \mathbb{Z}$

\[\text{if } c = bn \]
\[b = am \text{ for } n, m \in \mathbb{Z}\]
\[\text{then } c = (am)n\]
\[c = a(mn)\]

and $m \cdot n$ is an integer $\times$ an integer, which is another integer

thus $c = a \cdot l$ for $l \in \mathbb{Z}$

which follows the rule for divisible numbers $\square$
5. Show that \((\forall n \in \mathbb{N})\left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \in \mathbb{Z}\right)\).

Well, let's induct!

If \(n = 1\), \(\frac{(1)^3}{3} + \frac{(1)^2}{2} + \frac{1}{6} = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = \frac{2}{6} + \frac{3}{6} + \frac{1}{6} = 1\), which is an integer.

So our proposition is true for \(n = 1\).

Suppose it's true for \(k\), \(\frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6} \in \mathbb{Z}\) and we need to show that it is true for \(k+1\) that \(\frac{(k+1)^3}{3} + \frac{(k+1)^2}{2} + \frac{k+1}{6} \in \mathbb{Z}\).

\[
\frac{(k+1)^3}{3} + \frac{(k+1)^2}{2} + \frac{k+1}{6} = \frac{k^3 + 3k^2 + 3k + 1}{3} + \frac{k^2 + 2k + 1}{2} + \frac{k + 1}{6}
\]

\[
= \frac{k^3}{3} + k^2 + \frac{k}{2} + \frac{1}{6} + \frac{3k^2}{3} + \frac{3k}{3} + \frac{1}{3} + k^2 + \frac{2k}{2} + \frac{1}{2} + \frac{k}{6} + \frac{1}{6}
\]

\[
= \frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6} + 1 + k + k^2 + k
\]

Since 1, \(k^2\), and \(k\) are all integers and by our inductive hypothesis \(\frac{k^3}{3} + \frac{k^2}{2} + \frac{k}{6}\) is an integer, the sum of all these parts is an integer. Hence.

So our statement is true for all natural numbers \(n\). \(\Box\)