Examlet 1 Foundations of Advanced Math 2/3/06

1. a) State the definition of an odd integer.

an odd integer n can be written in the form $n = 2m+1$ for some integer m.

Yes

b) Suppose that n is an integer. Show that if n^2 is even, then n is even.

$n^2 = 2a$ for some integer a

n can either be even or odd.

- if n is even, it is in the form $n = 2b$ for some integer b

$n^2 = (2b)^2 = 4b^2 = 2(2b^2) \xrightarrow{\text{let}} 2b^2 = c$

$= 2c$ shows that an even squared is even

- if n is odd, it is in the form $n = 2d+1$ for some integer d.

$n^2 = (2d+1)^2 = 4d^2 + 4d + 1 = 2(2d^2+2d)+1 \xrightarrow{\text{let}} 2d^2+2d = e$

$= 2e+1$ shows ~~that~~ that an odd squared is odd.

- since these are the only two cases, the only way for n^2 to be even is if n is even.

Good.

2. a) Make a truth table for the statement $(P \wedge \neg Q) \Rightarrow R$.

P	Q	R	$\neg Q$	$(P \wedge \neg Q)$	$(P \wedge \neg Q) \Rightarrow R$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

Great

b) The propositional $\neg(P \Leftrightarrow Q)$ is equivalent to $\neg P \Leftrightarrow \neg Q$.

P	Q	$P \Leftrightarrow Q$	$\neg(P \Leftrightarrow Q)$	$\neg P$	$\neg Q$	$(\neg P \Leftrightarrow \neg Q)$
T	T	T	F	F	F	T
T	F	F	T	F	T	F
F	T	F	T	T	F	F
F	F	T	F	T	T	T
			*			*

Excellent

These are not equal. You can't distribute a $(\neg)$ sign.

$(\neg P \Leftrightarrow \neg Q)$ would be equal to $(P \Leftrightarrow Q)$ but that's not what it's asking.

3. $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is rational and therefore can be written in the form $\frac{a}{b}$ $a, b \in \mathbb{Z}$ and a + b don't have any common factors

So: $\sqrt{2} = \frac{a}{b}$

Square both sides: $2 = \frac{a^2}{b^2}$

rearrange to: $2b^2 = a^2$

Thus a^2 must be even making a also even because an odd^2 = odd. Then a can be written in the form $a = 2r$ $r \in \mathbb{Z}$

replacing the a with 2r: $2b^2 = 4r^2$

divide both sides by 2: $b^2 = 2r^2$

Hence b^2 must also be even forcing b to be even which contradicts the fact that a + b don't have common factors Thus proving $\sqrt{2}$ is irrational.

Wonderful

4. If c is divisible by b, and b is divisible by a, then c is divisible by a.

if c is divisible by b then $c = bn$ for $n \in \mathbb{Z}$

if b is divisible by a then $b = am$ for $m \in \mathbb{Z}$

Prove $c = ap$ for $p \in \mathbb{Z}$

if $c = bn$
$b = am$ for $n, m \in \mathbb{Z}$

then $c = (am)n$
$c = a(mn)$

and $m \cdot n$ is an integer $\times$ an integer, which is another integer

thus $c = a l$ for $l \in \mathbb{Z}$

which follows the rule for divisible numbers $\square$

Good

5. Show that $(\forall n \in \mathbb{N})\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6} \in \mathbb{Z}\right)$.

Well, let's induct!

If $\underline{n=1}$, $\frac{(1)^3}{3}+\frac{(1)^2}{2}+\frac{1}{6}=\frac{2}{6}+\frac{3}{6}+\frac{1}{6}=1$ which is an integer

So our proposition is true for $n=1$.

S'pose it's <u>true for</u> k, $\frac{k^3}{3}+\frac{k^2}{2}+\frac{k}{6} \in \mathbb{Z}$ and we need to show that it is <u>true for $k+1$</u> that $\frac{(k+1)^3}{3}+\frac{(k+1)^2}{2}+\frac{k+1}{6} \in \mathbb{Z}$

$$\frac{(k+1)^3}{3}+\frac{(k+1)^2}{2}+\frac{k+1}{6}=\frac{k^3+3k^2+3k+1}{3}+\frac{k^2+2k+1}{2}+\frac{k+1}{6}$$

$$=\frac{k^3}{3}+\frac{3k^2}{3}+\frac{3k}{3}+\frac{1}{3}+\frac{k^2}{2}+\frac{2k}{2}+\frac{1}{2}+\frac{k}{6}+\frac{1}{6}$$

$$=\frac{k^3}{3}+\frac{k^2}{2}+\frac{k}{6}+1+k^2+k$$

Since 1, k^2, k are all integers and by our inductive hypothesis $\frac{k^3}{3}+\frac{k^2}{2}+\frac{k}{6}$ is an integer, the sum of all these parts is an integer.

<u>Nice</u>.

So our statement is true for all natural numbers n. $\square$