Examlet 4 Foundations of Advanced Math 4/14/06

Each problem is worth 10 points. Appropriate justification is required for full credit.

1. a) Let R be a relation on the set A. State the definition of R being transitive.

$(\forall a,b,c \in A)\ [aRb \wedge bRc \rightarrow aRc]$

Good

b) Give an example of a relation on the set $\{a, b, c\}$ which is reflexive and symmetric, but not transitive.

$\{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$

They are all reflexive and Symmetric.

They are not transitive because:

$(a,b) \wedge (b,c) \rightarrow (a,c)$

which is not an element in our set.

Excellent!

2. Let $A = \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$. Let $R = \{(\heartsuit,\diamondsuit),(\spadesuit,\clubsuit)\}$. Is R reflexive? Symmetric? Transitive?

Reflexive? No. b/c none of the elements of A relate to themselves for R to be reflexive R would have to include $(\heartsuit,\heartsuit)$, $(\diamondsuit,\diamondsuit)$, $(\spadesuit,\spadesuit)$, and $(\clubsuit,\clubsuit)$ but R doesn't include these elements. Exactly.

Symmetric? No, R includes $(\heartsuit, \diamondsuit)$ but not $(\diamondsuit, \heartsuit)$ so R is not symmetric.

Yes!

Transitive? R is vacusouly transitive.

$(\heartsuit, \diamondsuit)$ but you don't have $\diamondsuit$ relating to something so the 1st condition of transitive is not met

the Same with $(\spadesuit, \clubsuit)$, $\clubsuit$ doesn't relate to something.

Wonderful.

3. If R and S are symmetric relations on a set A, then $R \cap S$ is a symmetric relation on A.

Proof: Suppose (a,b) is an arbitrary element in $R \cap S$.

So, by definition of intersection, $(a,b) \in R$ and $(a,b) \in S$.

Since we know that R is symmetric, we know that $(b,a) \in R$

And since S is symmetric, we know that $(b,a) \in S$.

By definition of intersection (since $(b,a) \in R \wedge (b,a) \in S$), $(b,a) \in R \cap S$

$\therefore$ whenever a set $R \cap S$ has an arbitrary element (a,b) and R and S are both symmetric, (b,a) is also an element of $R \cap S$ thus making $R \cap S$ symmetric. $\square$

Wonderful!

4. Define a relation ~ on the set of ordered pairs of real numbers by

$$(x_1, y_1) \sim (x_2, y_2) \text{ iff } \sqrt{x_1^{\,2} + y_1^{\,2}} = \sqrt{x_2^{\,2} + y_2^{\,2}}.$$

a) Find three points which are related to the point (2,0) under ~.

$$(-2,0) \sim (2,0) \quad \because \sqrt{(-2)^2 + (0)^2} = \sqrt{(2)^2 + (0)^2}$$

$$(0,2) \sim (2,0) \quad \because \sqrt{(0)^2 + (2)^2} = \sqrt{(2)^2 + (0)^2}$$

$$(0,-2) \sim (2,0) \quad \because \sqrt{(0)^2 + (-2)^2} = \sqrt{(2)^2 + (0)^2}$$

b) Is ~ an equivalence relation on $\mathbb{R} \times \mathbb{R}$?

Well, we have to check reflexive, symmetric, and transitive.

Reflexive: $(x,y) \sim (x,y) \quad \because \sqrt{(x)^2 + (y)^2} = \sqrt{(x)^2 + (y)^2}$

Symmetric: $(x_1, y_1) \sim (x_2, y_2) \Rightarrow \sqrt{(x_1)^2 + (y_1)^2} = \sqrt{(x_2)^2 + (y_2)^2}$

$$\Rightarrow \sqrt{(x_2)^2 + (y_2)^2} = \sqrt{(x_1)^2 + (y_1)^2}$$

By the symmetric property of =.

$$\Rightarrow (x_2, y_2) = (x_1, y_1)$$

Transitive: S'pose $(x_1, y_1) \sim (x_2, y_2)$ and $(x_2, y_2) \sim (x_3, y_3)$.

That means $\sqrt{(x_1)^2 + (y_1)^2} = \sqrt{(x_2)^2 + (y_2)^2}$ and $\sqrt{(x_2)^2 + (y_2)^2} = \sqrt{(x_3)^2 + (y_3)^2}$.

But then by the transitive property of =, we have

$$\sqrt{(x_1)^2 + (y_1)^2} = \sqrt{(x_3)^2 + (y_3)^2}$$

so $(x_1, y_1) \sim (x_3, y_3)$. Thus the relation is transitive.

So since ~ is reflexive, symmetric, and transitive, it's an equivalence relation. $\square$

5. a) Let $f \subseteq A \times B$ be a bijective function. Define f^{-1} in terms of ordered pairs.

$$f^{-1} = \{ (b,a) \mid (a,b) \in f \}$$

It's the set of ordered pairs from f, but reversed.

b) Let $f \subseteq \mathbb{R} \times \mathbb{R}$ and $g \subseteq \mathbb{R} \times \mathbb{R}$ be functions. Define $f + g$ in terms of ordered pairs.

$$f+g = \{ (a, b+c) \mid (a,b) \in f \wedge (a,c) \in g \}$$

For any given first element a, $f+g$ assigns the second element that's the sum of the corresponding second elements from f and g.