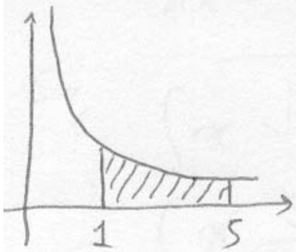
Exam 1 Calc 2 1/31/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Write an integral for the area of the region between the graph of $y = 1/x^2$ and the x-axis between $x = 1$ and $x = 5$.



lower limit = 1
upper limit = 5
for area, $\int_1^5 y\, dx$

$= \int_1^5 (1/x^2)\, dx$. Good

2. Find the average value of $f(x) = 1/x$ on the interval [1,5].

$AV = \frac{1}{b-a}\int_a^b f(x)\, dx$

$\frac{1}{5-1}\int_1^5 \frac{1}{x}\, dx$

$\frac{1}{4}\left(\ln x\right)\Big|_1^5$

$\frac{1}{4}\left(\ln(5) - \underbrace{\ln(1)}_{0}\right)$

$\boxed{\frac{1}{4}\left(\ln(5)\right)}$

Great

3. Evaluate $\int x\sqrt{9-x^2}\,dx$.

$$= \int x\sqrt{9-x^2}\,dx$$

let

$u = 9 - x^2$

$\frac{du}{dx} = \frac{d}{dx}(9-x^2)$

$\frac{du}{dx} = -2x$

or, $\frac{du}{-2x} = dx$

$$= \int x\sqrt{u}\,\frac{du}{-2x}$$

$$= -\frac{1}{2}\int u^{1/2}\,du$$

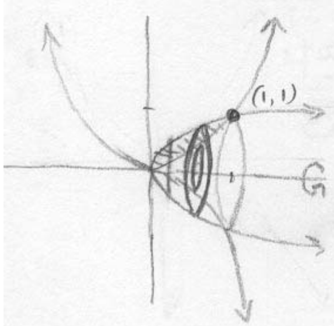
$$= -\tfrac{1}{2}\,\frac{u^{3/2}}{3/2} + C$$

$$= -\tfrac{1}{2}\times\frac{2}{3}\,(9-x^2)^{3/2} + C$$

$$= -\tfrac{1}{3}(9-x^2)^{3/2} + C$$

Excellent

4. Write an integral for the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y^2 = x$ around the x-axis.



volume by washer $= \int_a^b \pi\left((f(x))^2 - (g(x))^2\right)dx$

$g(x) = x^2$ intersect @ 1 & 0

$f(x) = \sqrt{x}$

$$\int_0^1 \pi\left((\sqrt{x})^2 - (x^2)^2\right)dx$$

$$\int_0^1 \pi(x - x^4)\,dx$$

Excellent

5. A spring has a natural length of 16 inches, and 15 pounds hold it stretched to a length of 20 inches. How much work, to the nearest tenth of a foot-pound, is done in stretching the spring from a length of 20 inches to a length of 22 inches?

natural length: 16 inches or 4/3 feet

$f = k \cdot x$ $\qquad 20'' - 16'' = 4'' \rightarrow \frac{1}{3}$ feet

$15\text{ lbs} = k\left(\frac{1}{3}\text{ ft}\right)$

$\boxed{45 \text{ lbs/ft} = k}$

$$W = \int_{1/3}^{1/2} (45x)\,dx$$

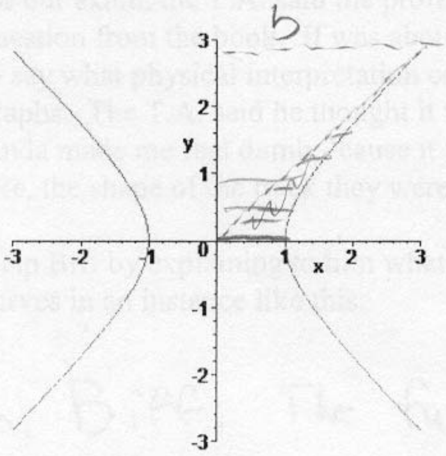
$$W = \left(\frac{45}{2}x^2\right)\Big|_{1/3}^{1/2}$$

$$W = \left(\frac{45}{8} - \frac{5}{2}\right)$$

Great

$\boxed{W = 3.1 \text{ ft} \cdot \text{lbs}}$

6. Write an integral for the area between the line $y = x$, the right half of the hyperbola $x^2 - y^2 = 1$, the line $y = 0$, and the line $y = b$ for some positive constant b.



$$\boxed{\int_0^b \sqrt{1+y^2}\,dy - \int_0^b y\,dy}$$

Integrate w/ respect to y b/c slices run horizontal, $x = \sqrt{1+y^2}$ is to the right of $x = y$, so $x = \sqrt{1+y^2}$ is the upper bound + $x = y$ is the lower bound.

Great

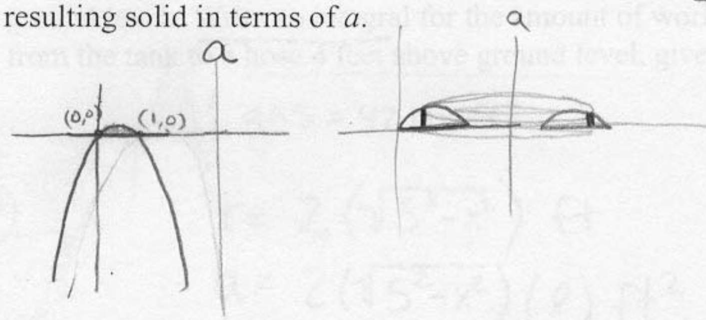
7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap, this Calc stuff is killin' me. In our discussion section when we were reviewing for our exam, the T.A. said the professor told him we were supposed to talk about this one question from the book. If was about, like, two runners were racing, and we were supposed to say what physical interpretation could be made of the area between their two velocity graphs. The T.A. said he thought it was obvious so we weren't going to talk about it. That kinda made me feel dumb, 'cause it wasn't obvious to me. I was wondering if maybe it was, like, the shape of the track they were racing on or something?"

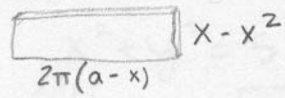
Help Biff by explaining to him what significance can be attached to the area between two curves in an instance like this.

If what's being graphed is velocity, then it is a graph of distance over time. So for one runner, at any point on the function is how much that runner ran at that certain time. So ~~total~~ total distance ran can be found by finding the area under that function. Since the graph has two runners, the area between the two would be the difference in distance run between them.

Good.

8. Suppose that the region between the graph of the parabola $y = x - x^2$ and the line $y = 0$ is rotated about the axis $x = a$ for some constant $a > 1$. Write an integral for the volume of the resulting solid in terms of a.





$h = x - x^2$

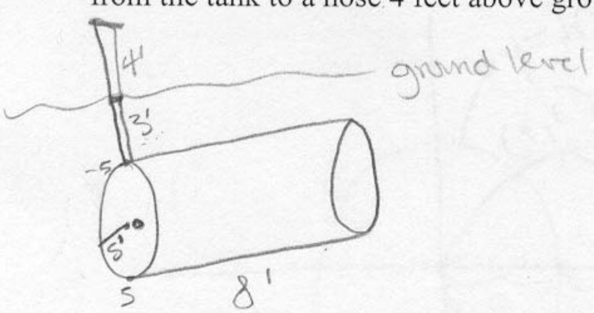
$r = (a - x)$

$\int_a^b 2\pi(\text{height})(\text{radius})\,dx$

$\int_0^1 2\pi(x - x^2)(a - x)\,dx$

Excellent

9. A gas pump draws from a tank shaped like a cylinder lying on its side (so that the flat faces are vertical) with a radius of 5 feet and length of 8 feet, and the top of the tank is 3 feet below ground level. Write an integral for the amount of work required to pump all the gasoline from the tank to a hose 4 feet above ground level, given that gasoline weighs 42 lbs/ft^3.



rectangle slices

area: $8(2\sqrt{25-x^2})$ ft^2

volume: $16\sqrt{25-x^2}\,\Delta x$ ft^3

force: $16\sqrt{25-x^2}\,\Delta x\,\cancel{\text{ft}^3} \cdot 42$ lbs/$\cancel{\text{ft}^3}$

Work: $672\sqrt{25-x^2}\,\Delta x\,(x+12)$

$x^2 + y^2 = 25$

$y = \sqrt{25-x^2}$

x, distance
(5, 17)
(0, 12)
(-5, 7)

$\frac{x+12}{4}$

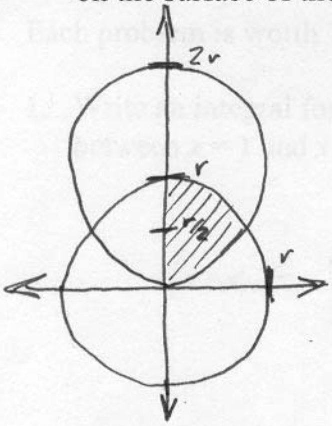
represents value that each x value must travel to top.

total work:

$$\boxed{\int_{-5}^{5}\left[(672\sqrt{25-x^2})(x+12)\right]dx}$$

Excellent

10. Find the volume common to two spheres, each with radius r, if the center of each sphere lies on the surface of the other sphere [Stewart 5th p. 454].



So we want to rotate the shaded region around the y-axis. But since it's symmetrical, we can do the top half (from $r/2$ to r on the lower sphere) and double it.

$$x^2+y^2=r^2$$

$$x^2=r^2-y^2$$

$$x=\pm\sqrt{r^2-y^2}$$

$$\text{Volume}_{Top} = \int_{r/2}^{r} \pi\left(\sqrt{r^2-y^2}\right)^2 dy$$

$$= \pi\int_{r/2}^{r}\left(r^2-y^2\right)dy$$

$$= \pi\left(r^2 \cdot y - \frac{1}{3}y^3\right)\Big|_{r/2}^{r}$$

$$= \pi\left[\left(r^3-\frac{r^3}{3}\right)-\left(\frac{r^3}{2}-\frac{r^3}{24}\right)\right]$$

$$= \pi\left(\frac{2}{3}r^3-\frac{11}{24}r^3\right)$$

$$= \frac{5\pi}{24}r^3$$

So $\text{Volume}_{Total} = 2\cdot\text{Volume}_{Top}$

$$= \frac{5\pi}{12}r^3$$