Exam 2 Calc 2 3/2/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Set up an integral for the arc length of the function $f(x) = \ln x$ from $(1,0)$ to $(e,1)$.

$$\int_1^e \sqrt{1+\frac{1}{x^2}}\,dx$$

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

b) Set up an integral for the surface area of the solid resulting when the portion of $f(x) = \ln x$ from $(1,0)$ to $(e,1)$ is rotated around the x-axis.

$$2\pi \int_1^e \ln x \cdot \sqrt{1+\frac{1}{x^2}}\,dx$$

Good

2. For the integral $\int_1^5 \sqrt{\ln x}\,dx$, the left-hand approximation with 2 subdivisions is 2.09629 and the right-hand approximation with 2 subdivisions is 4.63357. Find the trapezoidal and midpoint approximations with 2 subdivisions.

W

$L_2 = 2.09629$

$R_2 = 4.63357$

$T_2 = \frac{L_2 + R_2}{2} = \frac{2.09629 + 4.63357}{2}$

$T_2 = 3.36493$

$M_2 = f(2)\cdot 2 + f(4)\cdot 2 = 2\sqrt{\ln 2} + 2\sqrt{\ln 4}$

$M_2 = 4.019929$

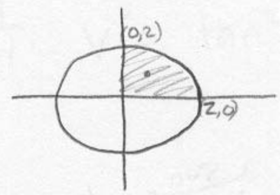
Great

3. Set up an integral for the mean of the p.d.f. $p(x)=\begin{cases}\frac{1}{3}e^{-\frac{x}{3}} & \text{for } x \geq 0\\ 0 & \text{for } x<0\end{cases}$.

$$\bar{x}=\int_{-\infty}^{\infty} x\cdot p(x)\,dx=\int_{-\infty}^{0} x\cdot 0\,dx+\int_{0}^{\infty} x\cdot\frac{1}{3}e^{-x/3}\,dx=\int_{0}^{\infty}\frac{x}{3}e^{-x/3}\,dx$$

4. Set up integrals for $\bar{x}$, the x coordinate of the center of mass of the first-quadrant portion of a circle of radius 2 centered at the origin.

$$\bar{x}=\frac{\int_a^b x\cdot f(x)\,dx}{\int_a^b f(x)\,dx} \leftarrow \text{(area under curve)}$$



area of a circle $=\pi r^2$

area of a $\cdot 4^{th}$ of a circle $=\pi\frac{r^2}{4}$ $r=2$

$\pi\frac{4}{4}=\pi$

$x^2+y^2=r^2$

$x^2+y^2=2^2$

$y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

$y=\sqrt{4-x^2}$ positive because 1^{st} quadrant

$$\bar{x}=\frac{\int_0^2 x\sqrt{4-x^2}\,dx}{\int_0^2\sqrt{4-x^2}\,dx}$$

OR

$$\bar{x}=\frac{\int_0^2 x\sqrt{4-x^2}\,dx}{\pi}$$

Excellent!

$\cos^2 x + \sin^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

5. Derive the integral formula $\int \tan^n u\,du = \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du + C$.

$\int \tan^n u\,du$ — * $\sec u$ is even so save a $\sec^2 u$, and convert a $\tan^2 u$.

$= \int \tan^{n-2} u(\underline{\sec^2 u - 1})\,du$

$= \int \tan^{n-2} u \sec^2 u\,du - \int \tan^{n-2} u\,du$

~~x~~-Substitution

let $x = \tan u$

$\frac{dx}{du} = \sec^2 u$

$\frac{dx}{\sec^2 u} = du$

$= \int x^{n-2} (\cancel{\sec^2 u}) \cdot \frac{dx}{\cancel{\sec^2 u}} - \int \tan^{n-2} u\,du$

$= \int x^{n-2}\,dx - \int \tan^{n-2} u\,du$

$= \frac{1}{n-1} x^{n-1} - \int \tan^{n-2} u\,du$

Excellent

$$\boxed{= \frac{1}{n-1}\tan^{n-1} u - \int \tan^{n-2} u\,du + C}$$

6. Evaluate $\displaystyle\int \frac{dx}{x\left(1+x^2\right)}$. Partial fractions

I wish: $\boxed{\dfrac{1}{x(1+x^2)} = \dfrac{A}{x} + \dfrac{Bx+C}{1+x^2}}\; x(1+x^2)$

$$1 = A(1+x^2) + (Bx+C)x$$

$$1 = A + Ax^2 + Bx^2 + Cx$$

coefficients: $1 = A$

x terms: $0 = C$

x^2 terms: $0 = A+B$, $0 = 1+B$, $-1 = B$

$$\int \frac{1}{x}\,dx + \int \frac{-x}{1+x^2}\,dx$$

let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$\frac{du}{2x} = dx$

$$\int \frac{1}{x}\,dx + \int \frac{-\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}}$$

$$\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{u}\,du$$

Well done

$$\boxed{\ln|x| - \tfrac{1}{2}\ln(1+x^2) + C}$$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmy*god*! This trig stuff is *so* hard! This one problem from our exam review sheet was *so* confusing. The answers said ½ tan squared, but I got ½ sec squared, but my friend who's really smart said maybe they were the same, but they don't look the same to me. How can you tell?"

Help Bunny by explaining to her whether such answers are actually equivalent, and how you can tell.

Well bunny, no need to worry. The way that you can tell if your answer is the same is by taking the derivative of both of them to see what you get.

What the answer said:

$f(x) = \frac{1}{2}\tan^2 x$ → also written as $\frac{1}{2}(\tan x)^2$

$\boxed{f'(x) = \tan x \cdot \sec^2 x}$

What you got:

$f(x) = \frac{1}{2}\sec^2 x$ → also written as $\frac{1}{2}(\sec x)^2$

$f'(x) = \sec x \cdot \sec x \tan x$

$\boxed{= \sec^2 x \tan x}$

Smart!

So you see bunny, the answers are equivalent. By taking the derivatives of your answer and the books (and applying the chain rule) in the end everything comes out to be an equivalent answer since both of the derivatives equal $\sec^2 x \tan x$. You can check your answers this way in the future if you have problems figuring out if your answers match the book. Good luck.

8. Derive the integral formula $\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}\left(a\cos bu + b\sin bu\right) + C$.

$$\int e^{au}\cos bu\,du = \frac{e^{au}}{a^2+b^2}\left(a\cos bu + b\sin bu\right) + C$$

$$= \int e^{au}\cos bu$$

$u = \cos bu \qquad v = \frac{e^{au}}{a}$

$u' = -b\sin bu \qquad v' = e^{au}$

$$= \frac{e^{au}}{a}\cos bu + \int \frac{e^{au}}{a}\, b\sin bu\,du$$

$$= \frac{e^{au}}{a}\cos bu + \frac{b}{a}\int e^{au}\sin bu\,du$$

$u = \sin bu \qquad v = \frac{e^{au}}{a}$

$u' = b\cos bu \qquad v' = e^{au}$

$$= \frac{e^{au}}{a}\cos bu + \frac{b}{a^2}\cdot \sin bu\, e^{au} - \frac{b^2}{a^2}\int e^{au}\cos bu\,du$$

$$= \frac{e^{au}}{a}\cos bu + \frac{b}{a^2}\sin bu\; e^{au} - \frac{b^2}{a^2}\int e^{au}\cos bu\,du$$

Wonderful!

$$\left(\frac{a^2+b^2}{a^2}\right)\int e^{au}\cos bu\,du = \frac{e^{au}}{a}\cos bu + \frac{be^{au}\sin bu}{a^2}$$

$$\int e^{au}\cos bu\,du = \frac{1\,e^{au}}{a^2+b^2}\left(a\cos bu + b\sin bu\right) + C$$

9. Evaluate the integral $\int_1^\infty \frac{1}{x^p}\,dx$, where p is a constant.

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty} \left.\frac{x^{-p+1}}{-p+1}\right|_1^b \qquad \text{(as long as } p \neq 1\text{)}$$

$$= \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}\right)$$

$$= \lim_{b\to\infty}\left(\frac{b^{1-p}}{1-p} + \frac{1}{p-1}\right)$$

$$= \begin{cases} 0 + \dfrac{1}{p-1} & \text{if } p > 1 \\ \text{Diverges} & \text{if } p \le 1 \end{cases}$$

10. Derive the integral formula $\displaystyle\int \frac{du}{\left(a^2-u^2\right)^{3/2}} = \frac{u}{a^2\sqrt{a^2-u^2}} + C$.

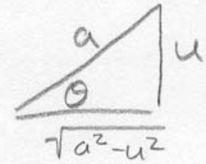
$$\int \frac{1}{(a^2-u^2)^{3/2}}\,du \qquad \underline{u = a\sin\theta} \qquad \underline{\frac{du}{d\theta} = a\cos\theta}$$

$$\int \frac{1}{\left(\sqrt{a^2-a^2\sin^2\theta}\right)^3} \cdot a\cos\theta\, d\theta$$

$$\int \frac{a\cos\theta}{\left(\sqrt{a^2(1-\sin^2\theta)}\right)^3}\, d\theta$$

$$\int \frac{a\cos\theta}{\left(\sqrt{a^2\cos^2\theta}\right)^3}\, d\theta = \int \frac{a\cos\theta}{(a\cos\theta)^3}\, d\theta$$

$$\int \frac{1}{a^2\cos^2\theta}\, d\theta = \frac{1}{a^2}\int \sec^2\theta\, d\theta$$



$$\underline{\frac{1}{a^2}\tan\theta + C}$$

$$\frac{1}{a^2}\left(\frac{u}{\sqrt{a^2-u^2}}\right) + C$$

Well Done!

$$\underline{\frac{u}{a^2\sqrt{a^2-u^2}} + C}$$