Exam 3 Calc 2 3/30/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Determine whether the sequence $\left\{\frac{2^n}{3^{n+1}}\right\}$ converges or diverges, and if it converges find its limit.

Converges.

$\frac{2^n}{3^{n+1}}$

$= \lim_{n\to\infty} \frac{2^n}{3^n \cdot 3} = \lim_{n\to\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n = \frac{1}{3} \lim_{n\to\infty}\left(\frac{2}{3}\right)^n \quad \left[\frac{2}{3}<1\right]$

$= \frac{1}{3}(0) = 0.$

Great

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}}$ converges or diverges, and if it converges find its limit.

Geometric $\sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^n$

When $n=1$: $\frac{2^1}{3^{1+1}} = \frac{2}{9}$

$n=2$: $\frac{2^2}{3^{2+1}} = \frac{4}{27}$

$r = \frac{4/27}{2/9} = 2/3$

$S = \frac{a}{1-r}$

$= \frac{2/9}{1-2/3}$

$= \frac{2/9}{1/3}$

$= 6/9$

$= 2/3$

3. Determine whether $y = x^2 - x^{-1}$ is a solution to the differential equation $xy' + y = 3x^2$.

$$y = x^2 - x^{-1}$$
$$y' = 2x + \frac{1}{x^2}$$

Now, substituting the value of 'y' & 'y'' in differential equation

$$x\left(2x + \frac{1}{x^2}\right) + x^2 - \frac{1}{x} = 3x^2$$

$$2x^2 + \cancel{\frac{1}{x}} + x^2 - \cancel{\frac{1}{x}} = 3x^2$$

Great

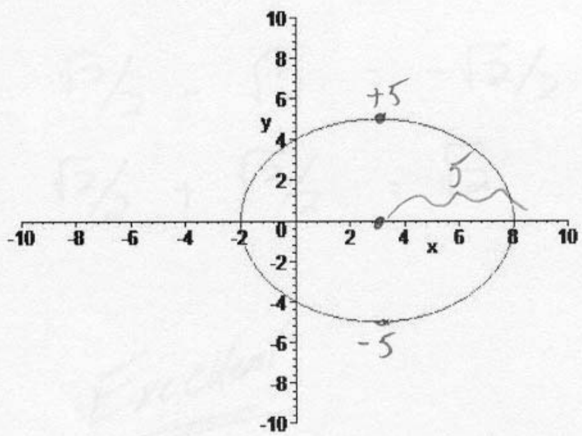
$$3x^2 = 3x^2$$

which is true

Therfore $y = x^2 - x^{-1}$ is a solution to the differential eqn.

4. Find an equation for the conic section shown

$$\frac{(x-3)^2}{25} + y^2/_{25} = 1$$



x-3 b/c shifted Right

/25 b/c both axis have r = 5

Great

5. Find the exact coordinates of the highest point on the graph with parametric equations

$$x = \cos t - 2\sin t,\ y = \sin t + \cos t.$$

highest point when slope $= 0$

$$\frac{dy}{dx} = \frac{\cos t - \sin t}{-\sin t - 2\cos t}$$ so when numerator $= 0$

$$\cos t - \sin t = 0$$

$$\cos t = \sin t$$

$$t = \frac{\pi}{4}$$

$$x = \cos\frac{\pi}{4} - 2\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} - \frac{2\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$

$$y = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$\left(-\frac{\sqrt{2}}{2}, \sqrt{2}\right)$$

Excellent

6. The amount of a particular pollutant (in grams) in a lake which is gradually being clogged with garbage is modeled by the differential equation $\frac{dp}{dt} = 5 - \frac{10p}{200-5t}$, where t is measured in years. Use Euler's method with steps of size $\Delta t = 10$ to approximate to the nearest tenth of a gram the amount of pollutant present after 20 years if $p(0) = 0$.

$P' = 5 - \frac{10P}{200-5t}$ step = 10

t	P	P'	ΔP
0	0	5	50
10	50	$5/3$	$50/3$
20	$200/3$		

$\frac{\Delta P}{10} \approx 5$ $\Delta P \approx 50$

$\frac{\Delta P}{10} \approx 5/3$ $\Delta P \approx 50/3$

$(0,0)$ $5 - \frac{0}{200} = 5$

$(10,50)$ $5 - \frac{500}{150} = \frac{5}{3}$

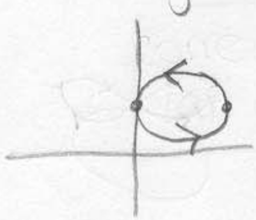
Good

66.7 grams

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Man! This math in college is so [bleep] hard! Our AP teacher in high school told us if you get a negative for area, you just take the negative away 'cause area's always positive, but I did that on our quiz about the parameter things and it got marked wrong. The professor even said after that in class that you couldn't just take off the negative, but I couldn't really get what he was saying you could do. I guess I'm screwed."

Help Biff by explaining to him what alternatives might be better than just removing negative signs or giving up.

It is true that area is always positive, but just taking it away could cause problems. One way to know if the area will be positive or negative is to watch your calculator graph the curve. If the line is moving from left to right, it will give you positive area, but if it is moving from right to left, you'll get negative area.



I try & break problems like this down in to pieces. In this graph for example, if you find the right-most limit & the left-most limit, you can divide the graph into 2 halfs, one with a positive area & one with a negative area. When solving it, the trick I always remember is that the limit that is on the left always goes on the bottom, + the limit on the right goes on the top. Then you don't have to worry about changing signs & things like that.

Good

8. Find the solution of the differential equation $\dfrac{dy}{dx} = y^2 + 1$ that satisfies the initial condition $y(0) = 2$.

by Separation of variables

$dy/dx = y^2 + 1$

$\frac{1}{y^2+1}\, dy = dx$

$\int \frac{1}{y^2+1}\, dy = \int dx$

$\tan^{-1}(y) = x + c$

$y = \tan(x + c)$

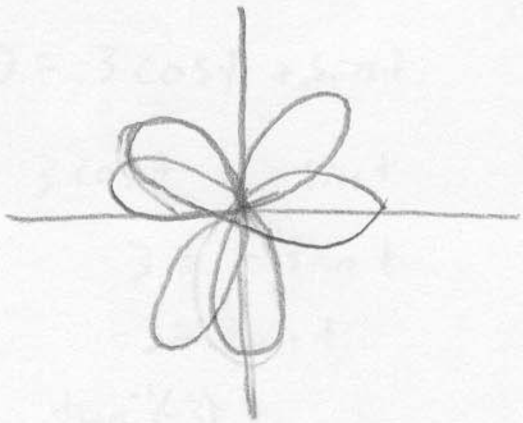
$2 = \tan(0 + c)$

$c = \tan^{-1}(2)$

$y = \tan(x + \tan^{-1}(2))$

Excellent

9. Find the exact area inside both $r = 2\cos 3\theta$ and $r = 2\sin 3\theta$.



$$2\cos 3\theta = 2\sin 3\theta$$

$$\cos 3\theta = \sin 3\theta$$

$$\underline{\theta = \frac{\pi}{12}}$$

$$\frac{1}{2}\int_0^{\frac{\pi}{12}} (2\sin 3\theta)^2\, d\theta$$

$$2\int_0^{\frac{\pi}{12}} \sin^2 3\theta\, d\theta \qquad u = 3\theta \quad \frac{du}{d\theta} = 3$$

$$\frac{2}{3}\int_{\theta=0}^{\frac{\pi}{12}=\theta} \sin^2 u\, du$$

$$\frac{2}{3}\left(\frac{1}{2}(3\theta) - \frac{1}{4}\sin(2(3\theta))\right)\Big|_0^{\frac{\pi}{12}}$$

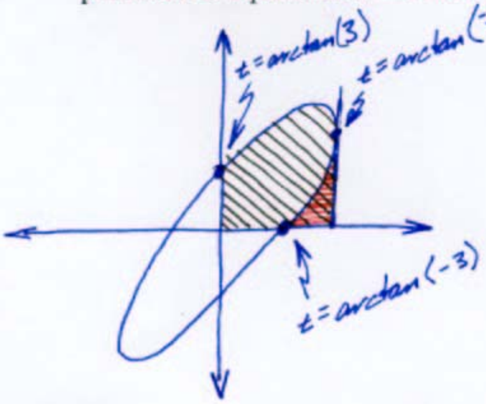
$$\frac{2}{3}\left(\left(\frac{\pi}{8} - \frac{1}{4}\cdot 1\right) - (0-0)\right)$$

$$\frac{2}{3}\left(\frac{\pi-2}{8}\right) \rightarrow \frac{2\pi-4}{24}$$ This is $\frac{1}{6}$th of the total

$$\frac{2\pi-4}{24}\cdot 6 = \frac{2\pi-4}{4} = \frac{\pi-2}{2}$$

Wonderful!

10. Set up an integral or integrals for the area of the first-quadrant portion of the curve with parametric equations $x = 3\cos t - \sin t$, $y = 3\cos t + \sin t$.



Crosses x-axis when:

$$0 = 3\cos t + \sin t$$

$$-3\cos t = \sin t$$

$$-3 = \tan t$$

$$t = \arctan(-3) \quad \text{(among others)}$$

Vertical tangent when denominator of $\frac{dy}{dx} = 0$:

$$0 = -3\sin t - \cos t$$

$$3\sin t = -\cos t$$

$$\tan t = -\tfrac{1}{3}$$

$$t = \arctan\left(\tfrac{-1}{3}\right) \quad \text{(among others)}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3\sin t + \cos t}{-3\sin t - \cos t}$$

Crosses y-axis when:

$$0 = 3\cos t - \sin t$$

$$\sin t = 3\cos t$$

$$\tan t = 3$$

$$t = \arctan(3)$$

$$\text{So Area} = \underbrace{\int_{\arctan(3)}^{\arctan\left(\frac{-1}{3}\right)} (3\cos t + \sin t)(-3\sin t - \cos t)\,dt}_{\text{Black shaded region}} - \int_{\arctan(-3)}^{\arctan\left(\frac{-1}{3}\right)} \underbrace{(3\cos t + \sin t)(-3\sin t - \cos t)\,dt}_{\text{Red shaded region}}$$