Exam 4　　Calc 2　　4/20/2007

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. What is the 4th degree MacLaurin polynomial for e^x? $F'(x) = e^x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

Good.

2. Give an example of a series which converges, but does not converge absolutely.

$\sum_{n=1}^{\infty} 1/n$ diverges by "p-series"

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by A.S.T

Good

3. Show that $\sum_{n=1}^{\infty} \frac{1}{n^3+2}$ converges. Comparison Test!

to $\sum 1/n^3$ which converges by P-series

$1/(n^3+2) \leq 1/n^3$

$n^3 \leq n^3+2$

$0 \leq 2$

Hidden

:)

I Know:

$0 \leq 2$ Kindergarden

$n^3 \leq n^3+2$ adding...

$1/(n^3+2) \leq 1/n^3$ division...

Excellent!

So: by comparison test $a_n \leq b_n$ & b_n is convergent then a_n also converges

4. Determine whether $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{n}{\ln n}$ converges or diverges.

Test for Divergence!

$\lim \left| (-1)^{n+1} \, n/\ln n \right| = 0$?

$\lim \left| n/\ln n \right|$ L'H $\lim 1/(1/n)$

$\lim_{n \to \infty} n = \infty$

Well done

So: the $\lim \neq 0$ so by test for Divergence the Series diverges

5. Determine whether $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges or diverges.

Let, $\sum_{n=1}^{\infty} n^2 e^{-n^3} = \int_1^{\infty} x^2 e^{-x^3}\,dx$

Using integral test.

$\lim_{a\to\infty} \int_1^a x^2 e^{-x^3}\,dx$

let $u = x^3$

$du = 3x^2\,dx$

$dx = \frac{du}{3x^2}$

$a, \lim_{a\to\infty} \int_{x=1}^{x=a} \frac{e^{-u}}{3}\,du$

$= \lim_{a\to\infty} -\frac{1}{3} e^{-u} \Big|_{x=1}^{x=a}$

$= \lim_{a\to\infty} -\frac{1}{3} e^{-x^3} \Big|_1^a$

$= \lim_{a\to\infty} -\frac{1}{3}\left(e^{-a^3} - e^{-1}\right)$

$= -\frac{1}{3}\left(e^{-\infty} - \frac{1}{e}\right)$

$= \frac{1}{3e}$

Nicely Done!

$\therefore \sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges because $\int_1^{\infty} x^2 e^{-x^3}\,dx$ converges

6. Find the radius of convergence of the series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$.

ratio test

$$\lim_{n\to\infty} \frac{\left|(-1)^{n+1} \frac{x^{2n+2}}{(2(n+1))!}\right|}{\left|(-1)^n \frac{x^{2n}}{(2n)!}\right|} \qquad \lim_{n\to\infty} \frac{\cancel{x^{2n}} \cdot x^2}{(2n+2)(2n+1)\cancel{(2n)!}} \cdot \frac{\cancel{(2n)!}}{\cancel{x^{2n}}}$$

$$\lim_{n\to\infty} \left|\frac{x^2}{(2n+2)(2n+1)}\right| \frac{x^2}{\infty} \to 0$$

the radius of convergence here is infinite. The intervals would be from $(-\infty, \infty)$ b/c the ratio test yielded zero.

Yes!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is *so* unfair. Our professor keeps saying there's more than one way to do one of these problems, right? Like, you know there's lots of different tests, but maybe there's two of them that both work on the same problem, right? So then on our quiz I got counted wrong for doing a different way from the professor. He said you had to use the Alternating Test, but instead I did it where you take off the negatives, right, like the absolute value of the series? So I showed that the absolute value of the series diverges, so the series has to diverge too, right? But he says I had to do it the Alternating Test way like he did, which is totally hypocritical if he says there's more than one way. I think I'm gonna see if Daddy will sue him."

   Help Bunny by explaining to her either what problems might exist with her approach, or how to defend it to her professor (short of litigation).

Bunny, Save your money you will lose this battle. An alternating Series can be solved numerous ways. You can prove that the series Converges by proving that it's absolute value converges but you can't prove it Diverges. An example of why not is $\sum \frac{(-1)^n}{n}$; this series converges by the A.S.T, but it's Absolute value $\sum 1/n$ diverges by the p-series, Your best bet any time you see the $(-1)^n$ in the problem is to try the A.S.T or the Ratio Test. Both of these tests will tell you if it increases or decreases.

Wonderful!

8. Find the 3[rd] degree Taylor Polynomial (centered at $a = 0$) for $f(x) = \tan x$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!} = x - \frac{x^2}{2!} + \frac{x^4}{5}$$

$f(x) = \tan x$ $\qquad f(0) = 0$

$f'(x) = \sec^2 x$ $\qquad f'(0) = 1$

$f''(x) = \frac{-2\cos x \sin x}{\cos^4 x}$ $\qquad f''(0) = 0$

$f'''(x) = \frac{-2(\sin^2 x + \cos^2 x)\cos^4 x + 4(\cos x)^3 \cdot \sin x \cdot 2\cos x \sin x}{\cos^8 x}$ $\qquad f'''(0) = 2$

$$x + \frac{2x^3}{3!}$$

Excellent

9. Use a 5[th] degree power series to give an approximation of ln 1.2. State your answer to five decimal places.

I know: $\frac{1}{1-x} \approx 1 + x + x^2 + x^3 + x^4$

So: $-\ln(1-x) \approx x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5}$

Or: $\ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$

Then substituting -0.2 in for x,

$$\ln(1+0.2) \approx 0.2 - \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} - \frac{(0.2)^4}{4} + \frac{(0.2)^5}{5}$$

$$\approx 0.18233$$

10. Use a Taylor polynomial of degree 9 for $\sin(x^3)$ to approximate $\int_0^{0.5} \sin(x^3)\,dx$ to eight decimal places.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin(x^3) = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!}$$

$$\sin(x^3) = x^3 - \frac{x^9}{3!} + \cancel{\frac{x^{15}}{5!}}$$

$$\int_0^{.5} \sin(x^3) = \int_0^{.5} x^3 - \frac{x^9}{3!}\,dx = \int_0^{.5} x^3 - \frac{x^9}{6}\,dx$$

$$\frac{x^4}{4} - \frac{x^{10}}{60} \Big|_0^{.5}$$

$$\left(\frac{(.5)^4}{4} - \frac{(.5)^{10}}{60}\right) - (0 - 0) \qquad \frac{1}{64} - 0.000016276 \approx 0.01560872$$

Excellent