Examlet 3a Foundations of Advanced Math 3/23/07

1. a) State the definition of a decreasing function

if the function f is decreasing, then $x > y \Rightarrow f(x) < f(y)$

Great

b) Give an example of a set which is not bounded.

$\mathbb{N}$ is not bounded because for any $M \in \mathbb{R}$, there is a natural number larger than M.

Excellent

2. Is the product of an even function with an odd function even or odd? Support your answer.

Proposition: The product of an even function with an odd function is odd.

Proof: Let f be an even function. So by definition $f(-x) = f(x)$.
Let g be an odd function. By definition then $g(-x) = -g(x)$.

The product of f and g would look as follows.

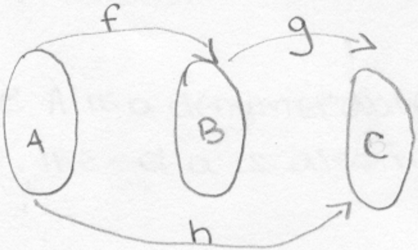
$(f * g)(-x) = f(-x) * g(-x)$. By our definitions we can write this as $f(x) * -g(x)$. We can ~~re~~write this equation as $(f * (-g))(x)$ which in turn can be written as $-(f * g)(x)$.

So $(f * g)(-x) = -(f * g)(x)$. This is the definition of an odd function. So the product of an even function and odd function is odd. $\square$

Nice!

3. Let $f: A \rightarrow B$ and $g: B \rightarrow C$ both be injections. Show that $g \circ f$ is injective.

injective = one-one



Take two elements from set A, call them a_1, a_2; let $h = (g \circ f)$

Suppose $h(a_1) = h(a_2)$

$(g \circ f)(a_1) = (g \circ f)(a_2)$

$g(f(a_1)) = g(f(a_2))$

Then, $f(a_1) = f(a_2)$ $\Rightarrow$ Because g is injective.

Then $a_1 = a_2$ $\Rightarrow$ Because f is injective.

Then h is injective since the only way $h(a_1) = h(a_2)$ is if $a_1 = a_2$, making every output unique for each input. $\square$

Excellent

4. Suppose that A is a denumerable set, and let $B = \{1,2,3\}$. Prove that $A \times B$ is denumerable, or prove that it isn't.

So $A \times B$ consists of ordered pairs whose first elements are from A and whose second elements are from B, so for each $a \in A$ we have $(a, 1) \in A \times B$, $(a, 2) \in A \times B$, and $(a, 3) \in A \times B$.

To show $A \times B$ denumerable we need to provide a bijection $f: \mathbb{N} \to A \times B$. But we know that A is denumerable so $\exists$ a bijection $g: \mathbb{N} \to A$. Then our basic idea will be to send the first throdd, throddodd, and threeven to elements $(a_1, 1)$, $(a_1, 2)$, and $(a_1, 3)$, where $a_1 = f(1)$, and so forth.

The actual bijection is

$$g(n) = \begin{cases} (f(k), 1) & \text{if } n = 3k-2 \text{ for some } k \in \mathbb{N} \\ (f(k), 2) & \text{if } n = 3k-1 \text{ for some } k \in \mathbb{N} \\ (f(k), 3) & \text{if } n = 3k \text{ for some } k \in \mathbb{N} \end{cases}$$

Every element of $A \times B$ is in the range of g because every element of A is in the range of f and every element of B is paired with each. The function is injective because no outputs from different branches have the same second element and f is injective.

5. Show that for any $a, b \in \mathbb{R}$ with $a < b$, $[0, 1]$ and $[a, b]$ are equipollent.

For two sets to be equipollent there must be a bijection between them.

So define $f: [0,1] \to [a,b]$ by $f(x) = (b-a)x + a$. We can see f is injective because it's an increasing function, and surjective is a straightforward problem from last year's examlet.

Scratch:

$(0,a)$ to $(1,b)$

$m = \frac{b-a}{1-0} = b-a$

$y - a = (b-a)(x-0)$

$y = (b-a)x + a$