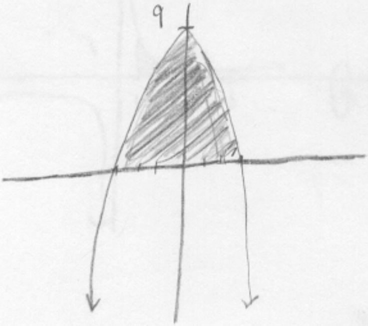
Exam 1 Calc 2 1/30/2008

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Write an integral for the area of the region between the graph of $y = 9 - x^2$ and the x-axis.



$y = 0$

$y = 9 - x^2$

$0 = 9 - x^2$

$0 = (x-3)(x+3)$

$\pm 3 = x$

Symmetry

$$\text{Area} = 2\int_0^3 (9-x^2) - (0)\, dx$$

$$= \boxed{2\int_0^3 (9-x^2)\, dx}$$

Great

2. Write an integral for the average value of the function $f(x) = x^3$ on the interval [0, 2].

$$\text{Avg value} = \frac{1}{b-a}\int_a^b f(x)\, dx$$

$$= \frac{1}{2-0}\int_0^2 x^3\, dx$$

$$= \boxed{\frac{1}{2}\int_0^2 x^3\, dx}$$

Excellent!

3. Write an integral for the volume of the solid of revolution obtained by rotating the region between $y = 1/x$, $y = 0$, $x = 1$, and $x = 5$ around the x-axis.

$Y = \frac{1}{x} \quad Y = 0 \quad X = 1 \quad X = 5$

limits of integration

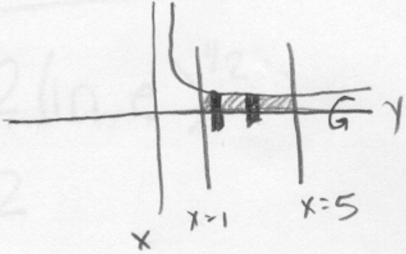
about x-axis

Use Volume by Discs

$$V_{Discs} = \pi \int_a^b [f(x)]^2 \, dx$$

Excellent!

$$\boxed{V = \pi \int_1^5 \left[\frac{1}{x}\right]^2 dx}$$



4. Suppose that a spring with a natural length of 6 inches requires 200 pounds to hold it stretched to a length of 9 inches. How much work is required to stretch it from a length of 6 inches to a length of 12 inches?

6 inches = 0.5 ft

9 inches = 0.75 ft

12 inches = 1 ft

We have:

$$200 = k.(0.75 - 0.5) = k.(0.25)$$

$$\rightarrow k = \frac{200}{0.25} = 800$$

$\rightarrow$ Force of the spring: $800x$

$\rightarrow$ Work to stretch it from 6 inches to 12 inches

$$W = \int_0^{0.5} 800x\,dx = 400x^2 \Big|_0^{0.5}$$

$$= 400.(0.25)$$

$$= 100 \text{ (ft - lbs)}$$

Well done

5. Evaluate $\displaystyle\int_e^{e^2} \frac{dx}{x\sqrt{\ln x}}$.

$u = \ln x$

$du = \frac{1}{x}\,dx$

$\ln e = 1$

$\ln e^2 = 2$

$$\int_1^2 \frac{1}{\sqrt{u}}\,du$$

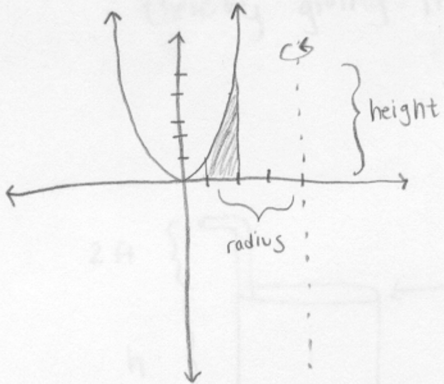
$$\int_1^2 u^{-1/2}\,du$$

$$\left.\frac{u^{1/2}}{1/2}\right]_1^2$$

$$\left.2\sqrt{u}\,\right]_1^2 \qquad 2\sqrt{2} - 2\sqrt{1} = \boxed{2\sqrt{2} - 2}$$

Nice Job!

6. Write an integral for the volume of the solid of revolution obtained by rotating the region between $y = x^2$, $y = 0$, $x = 1$, and $x = 2$ about the axis $x = 4$.



limits: $x = 1, 2$

Washers $\longrightarrow$ Volume $= \int 2\pi(\text{radius})(\text{height})\,dx$

$= \int_1^2 2\pi(4-x)(x^2)\,dx$

$= \boxed{2\pi \int_1^2 (4x^2 - x^3)\,dx}$

Excellent!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Well, crap. We're doing these work problems in calc, and they're really complicated, but I think there's an easier way. If I can figure out the weight for all the water in a tank, and it's gotta get up 2 feet to go out some spout, I can just take the whole weight times 2 feet and that's work, right? So all this stuff with integrals is just to make it hard for us to weed us out from pre-engineering, right?"

Help Biff by explaining to him why what he's proposing does or does not work.

Biff's theory would work if the tank was being lifted over the distance all at once by a crane, but the water is not being moved all at once, which means that it is being moved different distances. If the tank was 4 feet deep the water at the bottom has to travel 4 feet further, which requires more work, so because there is a changing amount of work done to move the water you have to use calculus and not just basic physics.

Wonderful!

8. Evaluate $\int x\sqrt{a+bx}\,dx$, where a and b are constants.

$a+bx = u \to x = \dfrac{u-a}{b}$

$\to \dfrac{du}{dx} = b$

$\to dx = \dfrac{du}{b}$

$$= \int \frac{u-a}{b}\sqrt{u}\,\frac{du}{b}$$

$$= \frac{1}{b^2}\int \left[(u-a)\sqrt{u}\right]du$$

$$= \frac{1}{b^2}\int \left[u^{3/2} - a.u^{1/2}\right]du$$

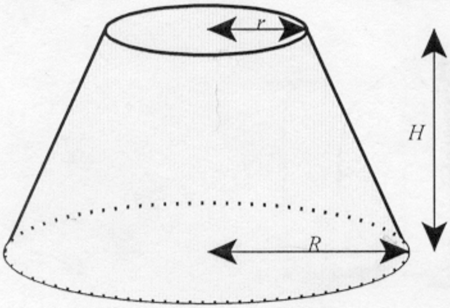
Great Job!

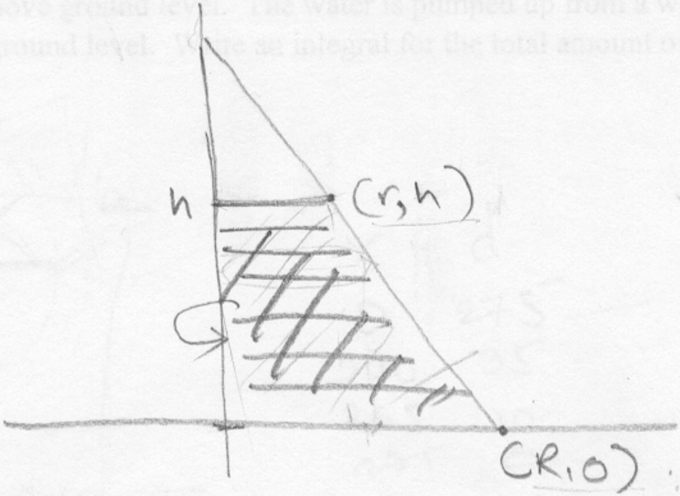
$$= \frac{1}{b^2}\left[\frac{2}{5}u^{5/2} - a\frac{2}{3}.u^{3/2}\right] + C$$

$$= \frac{1}{b^2}\cdot\frac{2}{5}(a+bx)^{5/2} - \frac{1}{b^2}\cdot\frac{2a}{3}(a+bx)^{3/2} + C$$

$$= \frac{2}{5b^2}\sqrt{(a+bx)^5} - \frac{2a}{3b^2}\sqrt{(a+bx)^3} + C$$

9. Write an integral for the volume of a frustum of a right circular cone with height h, lower base radius R, and top radius r. [Stewart]





Here, eqn of the line or, $(y-0) = (x-R)\left(\frac{h-0}{r-R}\right)$

or, $y = (x-R)\frac{h}{r-R}$.

or, $\frac{y(r-R)}{h} + R = x$.

So, rotating the shaded region around y axis gives us the volume

$$A = \pi\left(\frac{y(r-R)}{h} + R\right)^2$$

Excellent!

$$V = \pi\int_0^h \left(\frac{y(r-R)}{h} + R\right)^2$$

10. Jon plans to build a water tower on top of his house *just in case*. The holding tank will be a sphere 5 feet in radius, with its bottom 25 feet above ground level. The water is pumped up from a well which draws water from 240 feet below ground level. Write an integral for the total amount of work required to fill the holding tank.

$r = 5\text{ft}$

$x^2 + y^2 = 25$

radius $\rightarrow y = \sqrt{25 - x^2}$

Yes!

radius of slice $= \sqrt{25 - x^2}$

Area $= \pi\left(\sqrt{25 - x^2}\right)^2$

Volume $= \pi(25 - x^2)\,\Delta x$

Force $= 62.5\pi(25 - x^2)\,\Delta x$

Work $= 62.5\pi(25 - x^2)\,\Delta x \cdot (270 - x)$

$$\boxed{62.5\pi \int_{-5}^{5} (25 - x^2)(270 - x)\,dx}$$

Wonderful!