Exam 3 Calc 2 3/28/2008

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Find the first 4 partial sums of the series $\sum_{n=1}^{\infty} \frac{3}{10^n}$.

$S_1 = \frac{3}{10} = 0.3$; $S_2 = \frac{3}{10^1} + \frac{3}{10^2} = 0.3 + 0.03 = 0.33$

$S_3 = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333$; $S_4 = \frac{3}{10^1} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4}$

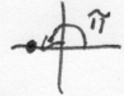
$= 0.3333$

b) Find the sum of the series in part a.

$\sum_{n=1}^{\infty} \frac{3}{10} \cdot \left(\frac{1}{10}\right)^{n-1} \Rightarrow a = \frac{3}{10}$, $r = \frac{1}{10}$ $(-1 < r < 1)$ Excellent!

$\rightarrow \sum_{n=1}^{\infty} \frac{3}{10^n} = \frac{\frac{3}{10}}{1 - \frac{1}{10}} = \frac{3}{10} : \frac{9}{10} = \frac{3}{10} \times \frac{10}{9} = \boxed{\frac{1}{3}}$

2. a) Convert the rectangular coordinates (–5, 0) to polar coordinates.



$x \quad y$

$$r = \sqrt{-5^2 + 0^2} = r = \sqrt{25} = 5 \qquad \theta = \pi$$

$$(r, \theta)$$

$$(5, \pi)$$

b) Convert the polar coordinates (6, π/3) to rectangular coordinates.

Great

$r \quad \theta$

$$x = 6\cos \pi/3 = 3$$

$$y = 6\sin \pi/3 = \frac{6\sqrt{3}}{2} = 3\sqrt{3}$$

$$x \quad y$$

$$(3, 3\sqrt{3})$$

3. Find an equation for the line tangent to the curve with parametric equations $x = t^4 + 1, y = t^3 + t$ at the point where $t = 2$.

$x = t^4 + 1$ $\quad\quad$ $y = t^3 + t$

$x' = 4t^3$ $\quad\quad$ $y' = 3t^2 + 1$

$$m = \frac{3t^2+1}{4t^3} = \frac{3(2)^2+1}{4(2)^3} = \frac{13}{32}$$

$x = (2)^4 + 1 = 17$

$y = (2)^3 + 2 = 10$

Nice!

$$\boxed{y - 10 = \frac{13}{32}(x - 17)}$$

4. Determine whether the sequence $a_n = \dfrac{3+5n^2}{n+n^2}$ converges or diverges, and if it converges find the limit.

Well, we know the sequence converges if the associated function does...

$$\lim_{x\to\infty} \frac{3+5x^2}{x+x^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{10x}{1+2x} \overset{L'H}{=} \lim_{x\to\infty} \frac{10}{2} = 5$$

So it converges.

5. A cup of coffee initially has a temperature of 95° C, and is left in a 20° C room. Suppose that you also know that the coffee will cool at a rate of 1° C each minute when its temperature is 70° C.

a) Write a differential equation for the temperature of the coffee after *t* minutes.

$$* \quad \frac{dT}{dt} = K(T-A) \rightarrow \frac{dT}{dt} = -1 = K(70-20)$$

$$(-)\ 50K = -1 \rightarrow K = \frac{-1}{50}$$

$$\rightarrow \boxed{\frac{dT}{dt} = \frac{-1}{50}(T-20)}$$

b) Use Euler's method with a step size of $\Delta t = 2$ minutes to approximate the temperature of the coffee after 4 minutes.

$T(0) = 95$ $\quad \Delta t = 2$

t	T	$\frac{dT}{dt}$	ΔT
0	95	-1.5	-3
2	92	-1.44	-2.88
4	$\boxed{89.12}$		

$\rightarrow T(4) = 89.12$

So after 4 minutes; the temperature of the coffee is 89.12

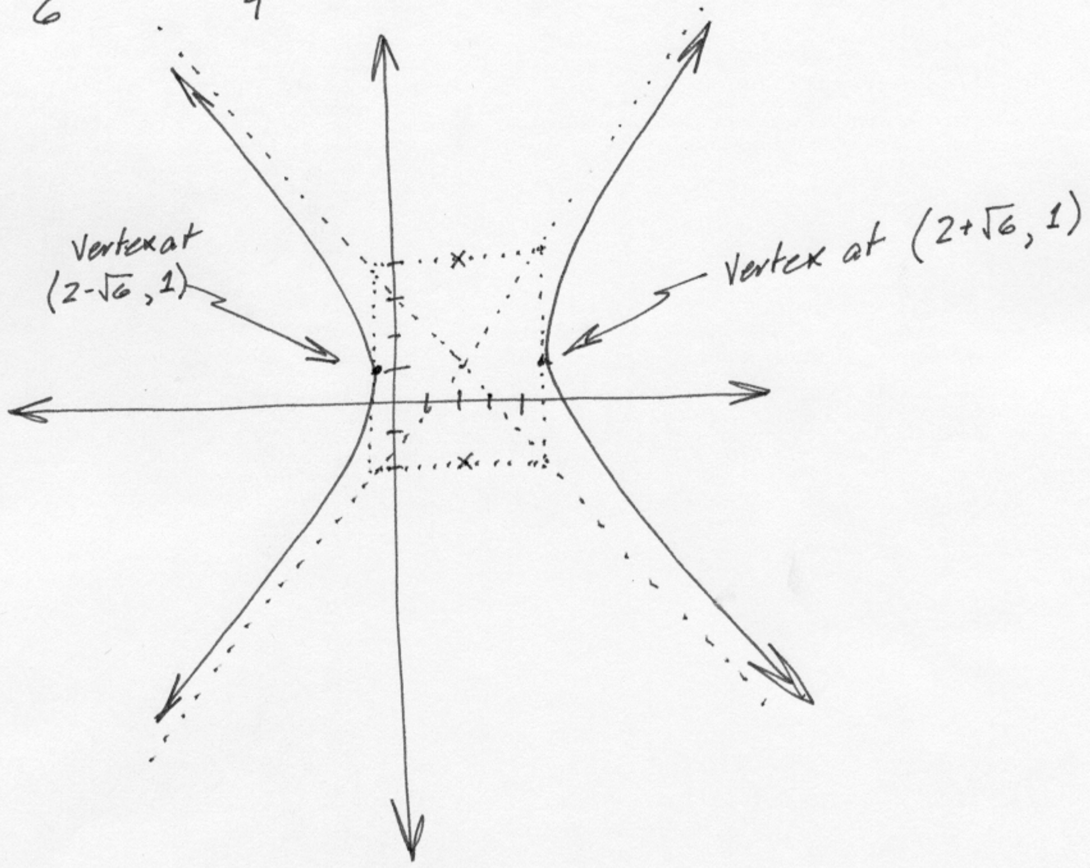
Excellent!

6. Identify the curve with equation $2y^2 - 3x^2 - 4y + 12x + 8 = 0$, and sketch a good graph of it.

$$2(y^2 - 2y + 1) - 3(x^2 - 4x + 4) = -8 + 2 - 12$$

$$\frac{2(y-1)^2}{-18} - \frac{3(x-2)^2}{-18} = \frac{-18}{-18}$$

$$\frac{(x-2)^2}{6} - \frac{(y-1)^2}{9} = 1$$ So it's a hyperbola



7. Biff is a calculus student at Enormous State University, and he has a question. Biff says "Dude, I love these parametric things, 'cause it's like all you gotta do is have your calculator graph 'em, you know? But for this one I think it screwed up somehow, 'cause it's $x = 3\cos t$ and $y = 3\sin t$, but the graph comes up like kind of a circle. That can't be right, 'cause trig stuff is all wavy, right?"

Help Biff by explaining what's going on.

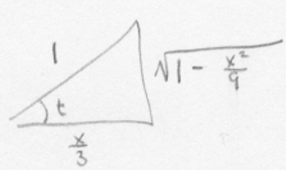
Parametric equations are two equations that, when graphed together, make one graph that could not be expressed by just one equation. By solving for t (the third variable, existing in both equations in the parametric) in one equation and substituting it into the other, the third variable is done away with and you end up with a different eqn entirely.

The graph isn't "all wavy" because when you carry out the substitution, the trig functions cancel out.

$x = 3\cos t$

$\frac{x}{3} = \cos t \Rightarrow \cos^{-1}\frac{x}{3} = t$

$y = 3\sin\left(\cos^{-1}\frac{x}{3}\right)$



$y = 3\sqrt{1-\frac{x^2}{9}}$

No more trig! cool!

Nice!

8. Find a solution to the differential equation $\frac{dH}{dt} = k\left(H - A\right)$ satisfying the initial condition $H(0) = H_0$.

$$\frac{dH}{dt} = k(H - A)$$

$$\rightarrow \int \frac{dH}{H-A} = \int k\,dt \qquad (A \leq H)$$

$$\rightarrow \ln|H - A| = kt + c_1$$

$$\rightarrow H - A = e^{kt + c_1} = e^{kt} \cdot \underbrace{e^{c_1}}_{}$$

C = constant

$$= C \cdot e^{kt}$$

$$\rightarrow H = C \cdot e^{kt} + A$$

So: if $H(0) = H_0$

$$\rightarrow H(0) = C \cdot 1 + A = H_0$$

$$\rightarrow C = H_0 - A.$$

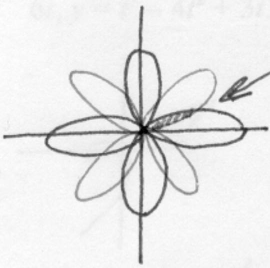
$$\rightarrow \boxed{H = (H_0 - A)e^{kt} + A}$$

Excellent

9. a) Find the area inside both of the curves r = cos 2θ and r = sin 2θ.

Because 8 making pieces

Because top/bottom symmetry of this piece



We want 8 of these

$$\text{So } A = 8 \cdot 2 \cdot \frac{1}{2}\int_0^{\pi/8} (\sin 2\theta)^2\, d\theta$$

let $u = 2\theta$, $\frac{du}{d\theta} = 2$, $\frac{du}{2} = d\theta$

$$= 8 \cdot \frac{1}{2}\int_0^{\pi/4} \sin^2 u\, du$$

$$= 4 \cdot \left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi/4}$$

$$= \left[2u - \sin 2u\right]_0^{\pi/4}$$

$$= \left(\frac{\pi}{2} - 1\right) - (0 - 2 \cdot 0)$$

$$= \frac{\pi}{2} - 1$$

Where do they cross?

$\cos 2\theta = \sin 2\theta$

$1 = \tan 2\theta$

$\frac{\pi}{4} = 2\theta$

$\theta = \frac{\pi}{8}$ is the first crossing.

Among others

b) Let n be a natural number, with $n \geq 2$. Find the area inside both of the curves r = cos nθ and r = sin nθ.

So it's like above, except there are $2n$ leaves for even n, each producing 2 areas of overlap, but there are n leaves with one area of overlap each for odd n.

We still need to know where that first crossing is, so:

$\cos n\theta = \sin n\theta$

$1 = \tan n\theta$

$\frac{\pi}{4} = n\theta$

$\theta = \frac{\pi}{4n}$ is the first crossing.

Among others

$$\text{So } A_1 = 2 \cdot \frac{1}{2}\int_0^{\frac{\pi}{4n}} (\sin n\theta)^2\, d\theta$$

For the area of one piece

let $u = n\theta$, $\frac{du}{d\theta} = n$, $\frac{du}{n} = d\theta$

$$= \frac{1}{n}\int_0^{\pi/4} \sin^2 u\, du$$

$$= \frac{1}{n} \cdot \left[\frac{u}{2} - \frac{1}{4}\sin 2u\right]_0^{\pi/4}$$

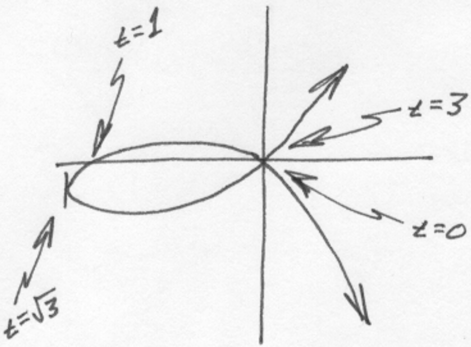
$$= \frac{1}{n} \cdot \left[\left(\frac{\pi}{8} - \frac{1}{4}\right) - (0 - 0)\right]$$

$$= \frac{\pi - 2}{8n}$$

And we have n such regions for odd values of n, but $4n$ such regions for even n, so

$$\text{Area} = \begin{cases} \frac{\pi - 2}{8n} \cdot n = \frac{\pi - 2}{8} & \text{for } n \text{ odd} \\ \frac{\pi - 2}{8n} \cdot 4n = \frac{\pi - 2}{2} & \text{for } n \text{ even} \end{cases}$$

10. Set up integrals for the area inside the loop of the graph of the function given parametrically by $x = t^3 - 9t$, $y = t^3 - 4t^2 + 3t$.



Just integrating from $t=0$ to $t=3$ isn't good since it's moving backwards and below the axis on parts of that, so we'll divide it up.

$$\text{Area} = \int_a^b y(t)\cdot x'(t)\,dt$$
$$= \int_a^b (t^3-4t^2+3t)(3t^2-9)\,dt$$
$$= \int_a^b (3t^5-12t^4+9t^3-9t^3+36t^2-27t)\,dt$$
$$= \int_a^b (3t^5-12t^4+36t^2-27t)\,dt$$

Crosses x axis when:

$$0 = y = t^3-4t^2+3t$$
$$0 = t(t^2-4t+3)$$
$$0 = t(t-3)(t-1)$$
$$t=0,\ t=3,\ t=1$$

Vertical tangent when:

$$0 = \frac{dx}{dt} = 3t^2-9$$
$$0 = 3(t^2-3)$$
$$t = \pm\sqrt{3}$$

So we'll set it up as

$$\text{Area} = -\int_0^1 (3t^5-12t^4+36t^2-27t)\,dt$$
$$-\int_1^{\sqrt{3}} (3t^5-12t^4+36t^2-27t)\,dt$$
$$-\int_{\sqrt{3}}^3 (3t^5-12t^4+36t^2-27t)\,dt$$

∵ backwards

∵ not included

∵ upside down