Exam 1 Differential Equations 2/8/08

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Determine whether the function $y = \sin t$ is a solution to the differential equation

$$\frac{d^2 y}{dt^2} + y = \sin t .$$

$y = \sin t$

$\frac{dy}{dt} = \cos t$

$\frac{d^2y}{dt^2} = -\sin t$

$-\sin(t) + \sin(t) \stackrel{?}{=} \sin(t)$

$0 \neq \sin(t)$

The 2 sides of the equation are not equal and therefore $y = \sin t$ is not a solution.

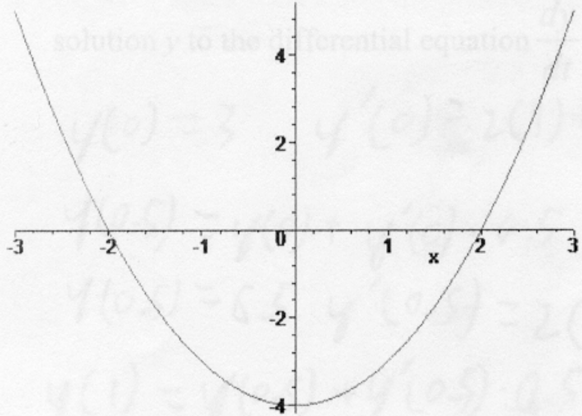
Good

2. State the definition of a separable differential equation.

A SEPARABLE DIFFERENTIAL EQUATION IS ONE WHICH CAN BE WRITTEN IN THE FORM $\frac{dy}{dt} = f(y)g(t)$ WHERE f IS A FUNCTION ONLY OF y AND g IS A FUNCTION ONLY OF t.

Excellent!

3. Sketch the phase line for the differential equation $dy/dt = f(y)$ if $f(y)$ has the graph shown:



$f = 0$ at -2 and 2, so these are equilibria.

$f > 0$ for $y < -2$, $y > 2$, so y is increasing there

$f < 0$ for $-2 < y < 2$, so y is decreasing there.

Great!

4. Find a general solution to the differential equation $\dfrac{dy}{dt} = t + t\,y^2$.

$$\frac{dy}{dt} = t(1+y^2)$$

separate variables: $\dfrac{1}{1+y^2}\,dy = t\,dt$

integrate: $\displaystyle\int \frac{1}{1+y^2}\,dy = \int t\,dt$

$$\arctan y = \tfrac{1}{2}t^2 + c$$

where c is some constant

$$y = \tan\left(\tfrac{1}{2}t^2 + c\right)$$

Great!

5. Use Euler's method with step size $\Delta t = 0.5$ to approximate $y(1)$ to the nearest hundredth for a solution y to the differential equation $\frac{dy}{dt} = 2y + 1$ subject to the initial condition $y(0) = 3$.

t	$\frac{dy}{dt}$	y	Δy
0	7.00	3.00	3.50
.50	14.00	6.50	7.00
1.00	28.00	13.50	14.00

$0 \Rightarrow 2(3)+1 = 6+1 = 7$

$\frac{\Delta y}{.5} = 7$

$\Delta y = 3.5$

$.5 \Rightarrow 2(6.5)+1 = 13+1 = 14$

$\frac{\Delta y}{.5} = 14$

$1 \Rightarrow 2(13.5)+1 = 27+1 = 28$

$\frac{\Delta y}{.5} = 28$

$y(1) = 13.50$

Good

6. Find a general solution to the differential equation $\frac{dy}{dt} = \frac{y}{t} + 4$.

$$\frac{dy}{dt} = \frac{y}{t} + 4$$

$$\frac{dy}{dt} - \frac{y}{t} = 4$$

$$\mu(t) = e^{\int -\frac{1}{t}\,dt} = e^{-\ln t} = e^{\ln t^{-1}} = \frac{1}{t}$$

$$\frac{1}{t}\frac{dy}{dt} - \frac{y}{t^2} = \frac{4}{t}$$

$$\frac{d}{dt}\left(\frac{1}{t} \cdot y\right) = \frac{4}{t}$$

— integrate —

$$\frac{y}{t} = 4\ln t + c$$ where c is some constant

$$y = 4t\ln t + ct$$

Well done!

7. Suppose that $\frac{dy}{dt} = f(y)$ is a differential equation satisfying the hypotheses of our existence and uniqueness theorems. Further suppose that $y_1(t) = 0$, $y_2(t) = 20$, and $y_3(t) = 30$ are all solutions for all t. If you're seeking a solution satisfying the initial condition $y(0) = 5$, what can you conclude about that solution?

By the existence theorem a solution must exist for $y(0) = 5$, and by the uniqueness theorem no solution may cross over an equilibrium solution, because that would mean there are multiple solutions at the point where they cross. So $y(t)$ is doomed to spend all eternity trapped in its miserable cage between $y=0$ and $y=20$, the nearest equilibria on either side of $y=5$. However, we can draw no conclusions about which equilibrium $y(t)$ will try in vain to break through. $y(t)$ wonders if the uniqueness theorem will allow it to dig under the equilibria. But that would be a job for multi-variable differential equations.

Excellent!

Or a new reality television series, if the writers' strike doesn't end soon!

8. Find the power series expansion for the general solution up to degree four to the differential equation $\frac{d^2y}{dt^2} + y = \sin t$.

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + a_6 t^6$$

$$y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 + 6a_6 t^5$$

$$y'' = 2a_2 + 6a_3 t + 12a_4 t^2 + 20a_5 t^3 + 30a_6 t^4$$

$$\sin(t) = \sum_{n=0}^{\infty} \frac{t^{(2n+1)}}{(2n+1)!}(-1)^n = t - \frac{t^3}{6} + \frac{t^5}{120}$$

$$2a_2 + a_0 = 0;\ 2a_2 = -a_0;\ a_2 = \frac{-a_0}{2}$$

$$6a_3 + a_1 = 1;\ 6a_3 = 1 - a_1;\ a_3 = \frac{1-a_1}{6}$$

$$12a_4 + a_2 = 0;\ a_4 = \frac{-a_2}{12} = \frac{a_0}{24}$$

$$20a_5 + a_3 = \frac{-1}{6};\ 20a_5 = \frac{-1}{6} + a_3;\ a_5 = \frac{-1}{120} - \frac{a_3}{20} = \frac{-1}{120} - \left(\frac{1-a_1}{120}\right)$$

$$30a_6 + a_4 = 0;\ a_6 = \frac{-a_4}{30} = \frac{-a_0}{720}$$

$$y = a_0 + \frac{1-a_1}{a_1} t - \frac{a_0}{2} t^2 + \left(\frac{1-a_1}{6}\right) t^3 + \frac{a_0}{24} t^4$$

Excellent

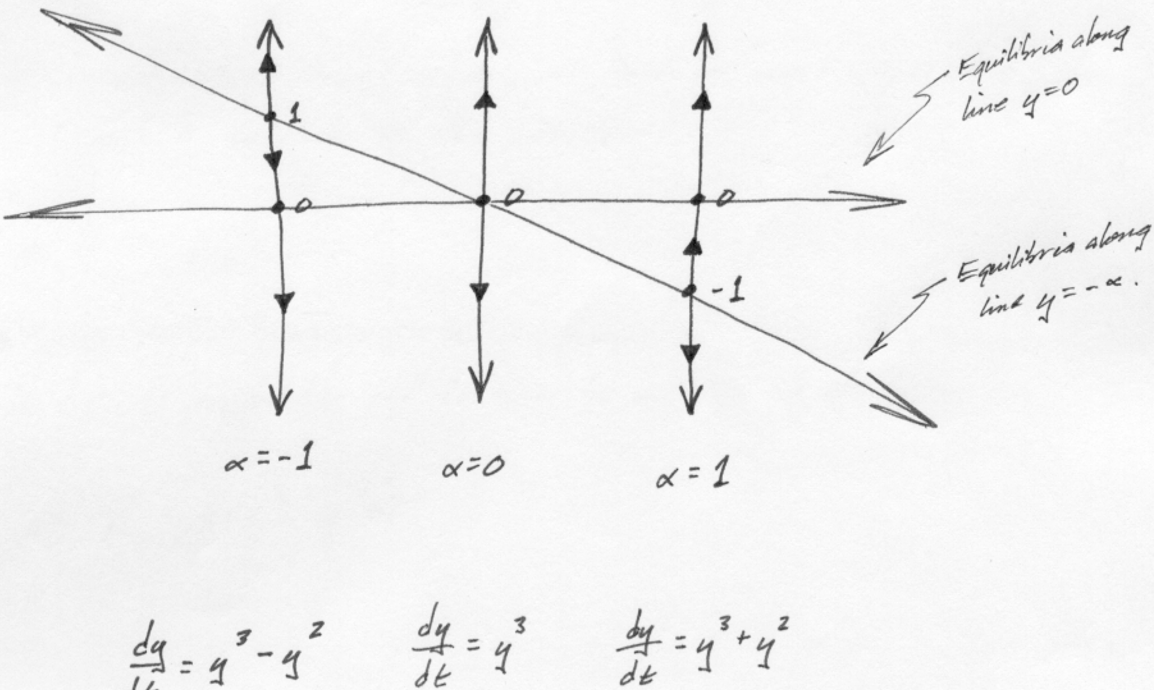
9. **Sketch the bifurcation diagram** for the differential equation $\frac{dy}{dt} = y^3 + \alpha y^2$. Include direction arrows on the phase lines and make clear the exact α values where bifurcations occur.

Find equilibria:

$$0 = y^3 + \alpha y^2$$

$$0 = y^2(y+\alpha)$$

$$\therefore\ y = 0 \text{ or } y = -\alpha$$



$$\frac{dy}{dt} = y^3 - y^2 \qquad \frac{dy}{dt} = y^3 \qquad \frac{dy}{dt} = y^3 + y^2$$