Foundations Induction Examples 1/23/08

Proposition: The product of any two consecutive natural numbers is even.

Proof: Well, let's proceed by induction to prove that the statement "n times $n + 1$ is even" holds for all natural numbers n. Suppose that the first integer is 1, so the second is 2. Then $1 \times 2 = 2 = 2(1)$ is even since it's 2 times an integer.

Now s'pose the statement is true for n, so that $n(n + 1) = 2m$ for some integer m, and we need to show that $n + 1$ times $n + 2$ is even. But

$$\begin{aligned}(n+1)(n+2) &= n^2 + 3n + 2 \\ &= (n^2 + n) + (2n + 2) \\ &= 2m + 2(n+1) && \text{[by our inductive hypothesis]} \\ &= 2(m + n + 1).\end{aligned}$$

So since $m + n + 1$ is an integer, we see that $(n + 1)(n + 2)$ is even. {Then since the statement has been shown true for $n = 1$, and since whenever the statement is true for n it is also true for $n + 1$, we can conclude by mathematical induction that the statement holds true for all natural numbers n. ☐}

It's perfectly acceptable to abbreviate the entire passage in braces above as "So by induction the statement holds for all natural numbers n. ☐"

Def.: If C is a collection of real numbers, we say b is an upper bound for C iff $(\forall x \in \text{C})\ b \geq x$.

Proposition: Any collection of exactly n distinct real numbers (where n is a natural number) has an upper bound.

Proof: Well, let's proceed by induction. Let C be a collection with just one real number in it, and call that number x. Then x itself is an upper bound for C, since $(\forall y \in \text{C})\ x \geq y$.

Now s'pose C is a collection with exactly two distinct real numbers in it, and call them x and y. Then either $x \geq y$ or $y \geq x$. In the first case x will be an upper bound for C, since $x \geq x$ and $x \geq y$, and similarly in the second case y is an upper bound for C.

Finally, suppose that any collection with exactly n distinct real numbers in it has an upper bound, and let D be a collection with exactly $n + 1$ real numbers. Let's first create a new collection C by taking all of the elements of D except one (label as a that element of D which was omitted from C). Then we know by our inductive hypothesis that C has an upper bound, call it b. Then either $a \geq b$ or $b \geq a$. Thus by the transitive property in the first case a is an upper bound for D, and in the second case b is. So by induction, we've shown that any collection of exactly n distinct real numbers has an upper bound. ☐