Examlet 4 **Foundations of Advanced Math** **4/11/08**

1. a) State the definition of a transitive relation.

$(\forall a, b, c \in A)[aRb \wedge bRc \rightarrow aRc]$

Yes.

b) Give an example of a relation on the set {1, 2, 3} which is reflexive but not symmetric.

$\{(1,1)(2,2)(3,3),(1,3)\}$

If symmetric would have $(3,1)$.

Great!

2. a) Suppose that ≡ is the relation on the set $A = \{a, b, c, d, e\}$ defined by $\equiv\, = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c), (d,d), (e,e)\}$. Write the equivalence classes corresponding to ≡ out explicitly.

$[a] = \{a, b, c\}$
$[b] = \{a, b, c\}$
$[c] = \{a, b, c\}$
$[d] = \{d\}$
$[e] = \{e\}$

Great!

$[a] = [b] = [c] = \{a, b, c\}$
$[d] = \{d\}$
$[e] = \{e\}$

b) Suppose that P is the partition $\{\{a\}, \{b, d\}, \{c, e\}\}$ of the set $A = \{a, b, c, d, e\}$. Find the relation R corresponding to P.

$R = \{(a,a), (b,b), (b,d), (d,b), (d,d), (c,c), (c,e), (e,c), (e,e)\}$

Exactly.

3. Let R be a relation on $\mathbb{Z}$ defined by $x\ R\ y \Leftrightarrow y \neq 5$. Determine whether R is reflexive, symmetric, or transitive, and support your conclusions well.

Reflexive? No. $5 \in \mathbb{Z}$, and if we set $x = 5$,

then xRx is false, since 5 is now

in the second position.

A relation must be reflexive for all of the elements of the set it's on.

Symmetric? No.

$x = 5 \quad y = 2 \rightarrow xRy$ because $y \neq 5$

However, y is not related to x, because $x = 5$

Transitive? Yes.

Let's take arbitrary elements a, b, and c.

$aRb \Leftrightarrow b \neq 5$

$bRc \Leftrightarrow c \neq 5$

In order for aRc to work, the only requirement is that $c \neq 5$. This has already been established, though.

Thus $(\forall a, b, c \in \mathbb{Z})[aRb \wedge bRc \Rightarrow aRc]$ and the relation is thus transitive. $\square$

Excellent

4. Let m be a natural number, and let $\equiv_m = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a - b = km \text{ for some } k \in \mathbb{Z}\}$. Show that $\equiv_m$ is an equivalence relation (this relation is usually called *congruence modulo m*).

$\equiv_m$ is reflexive because $a-a=0$, so $a-a=0 \cdot m$ + thus $(a,a) \in \equiv_m$ for any $m \in \mathbb{N}$.

$\equiv_m$ is symmetric because if $a-b=km$, $b-a=-km$. Since each k's negative will be an integer if k is an integer, $(a,b) \in \equiv_m \Rightarrow (b,a) \in \equiv_m$.

$\equiv_m$ is transitive because if $a-b=km$ and $b-c=jm$, we can say that $(a-b)+(b-c) = km+jm$. Therefore $a-c=(k+j)m$. Since the sum of two integers will be an integer, $(a,c) \in \equiv_m$. Thus, $[(a,b) \in \equiv_m \wedge (b,c) \in \equiv_m] \Rightarrow (a,c) \in \equiv_m$.

Since $\equiv_m$ is reflexive, symmetric, + transitive, it is an equivalence relation.

Beautiful.

5. a) Regarding the function $f: A \to B$ as a subset of $A \times B$, write the definition of f being onto.

f is onto $\iff$ $\forall b \in B\ \exists a \in A$ such that $(a, b) \in f$.

b) Recall χ_B, the characteristic function of a set B, from chapter 4; it was defined then by

$$\chi_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

Let A be a set and B be a subset of A. Write χ_B as a subset of $A \times \{0,1\}$.

$$\chi_B = \underbrace{B \times \{1\}}_{\substack{\text{This gets ordered} \\ \text{pairs } (b,1) \text{ for} \\ \text{every } b \in B.}} \cup \underbrace{(A - B) \times \{0\}}_{\substack{\text{This gets ordered} \\ \text{pairs } (a,0) \text{ for} \\ \text{every } a \text{ that's in} \\ A \text{ but not } B.}}$$