Homework 6 Foundations 4/16/08

Suppose that N is a set. We call N a **Peano system** iff the following conditions are satisfied:

I. $0 \in N$.

II. For each $x \in N$, there is a unique element $x' \in N$ (we call x' the *successor* of x).

III. $\forall x \in N, x' \neq 0$.

IV. $\forall x, y \in N, x' = y' \Rightarrow x = y$

V. If $M \subseteq N$ for which $0 \in M$ and $\forall x \in M, x' \in M$, then $M = N$.

Given a Peano system N, we make the following definition:

- Given $x, y \in N$, define their **sum** $x + y$ by
  - $x + 0 = x$
  - $x + (y') = (x + y)'$

Prove the following statements, given that N is a Peano system.

1. $\forall x, y \in N, x + (y + 0) = (x + y) + 0$.
2. $\forall x, y, z \in N, x + (y + z) = (x + y) + z \Rightarrow x + (y + z') = (x + y) + z'$.
3. $\forall x, y, z \in N, x + (y + z) = (x + y) + z$.
4. $0 + 0 = 0$.
5. $\forall y \in N, 0 + y = y \Rightarrow 0 + y' = y'$.
6. $\forall y \in N, 0 + y = y$.
7. $\forall x \in N, x' + 0 = (x + 0)'$.
8. $\forall x, y \in N, x' + y = (x + y)' \Rightarrow x' + y' = (x + y')'$.
9. $\forall x, y \in N, x' + y = (x + y)'$.
10. $\forall y \in N, 0 + y = y + 0$.
11. $\forall x, y \in N, x + y = y + x \Rightarrow x' + y = y + x'$.
12. $\forall x, y \in N, x + y = y + x$.
13. $\forall y \in N$, with $y \neq 0$, $0 \neq 0 + y$.
14. $\forall x, y \in N$, with $y \neq 0$, $x \neq x + y \Rightarrow x' \neq x' + y$.
15. $\forall x, y \in N$, with $y \neq 0$, $x \neq x + y$.
16. $\forall y, z \in N, 0 + y = 0 + z \Rightarrow y = z$.
17. $\forall x, y, z \in N, (x + y = x + z \Rightarrow y = z) \Rightarrow (x' + y = x' + z \Rightarrow y = z)$.
18. $\forall x, y, z \in N, x + y = x + z \Rightarrow y = z$.

Given a Peano system N, with the convention that $0' = 1$,we make the following definition:

- Given $x, y \in N$, define their **product** $x \cdot y$ by
  - $x \cdot 0 = 0$
  - $x \cdot (y') = (x \cdot y) + x$

Prove the following statements, given that N is a Peano system.

19. $\forall x, y \in N, x \cdot (y + 0) = x \cdot y + x \cdot 0.$

20. $\forall x, y, z \in N, x \cdot (y + z) = x \cdot y + x \cdot z \Rightarrow x \cdot (y + z') = x \cdot y + x \cdot z'.$

21. $\forall x, y, z \in N, x \cdot (y + z) = x \cdot y + x \cdot z.$

22. $\forall x, y \in N, x \cdot (y \cdot 0) = (x \cdot y) \cdot 0.$

23. $\forall x, y, z \in N, x \cdot (y \cdot z) = (x \cdot y) \cdot z \Rightarrow x \cdot (y \cdot z') = (x \cdot y) \cdot z'.$

24. $\forall x, y, z \in N, x \cdot (y \cdot z) = (x \cdot y) \cdot z.$

25. $0 \cdot 0 = 0.$

26. $\forall y \in N, 0 \cdot y = 0 \Rightarrow 0 \cdot y' = 0.$

27. $\forall y \in N, 0 \cdot y = 0.$

28. $\forall x \in N, x' \cdot 0 = 0.$

29. $\forall x, y \in N, x' \cdot y = x \cdot y + y \Rightarrow x' \cdot y' = x \cdot y' + y'.$

30. $\forall x, y \in N, x' \cdot y = x \cdot y + y.$

31. $\forall y \in N, 0 \cdot y = y \cdot 0.$

32. $\forall x, y \in N, x \cdot y = y \cdot x \Rightarrow x' \cdot y = y \cdot x'.$

33. $\forall x, y \in N, x \cdot y = y \cdot x.$

34. $\forall y \in N$, with $y \neq 1, 1 \neq 1 \cdot y.$

35. $\forall x, y \in N$, with $y \neq 1, x \neq x \cdot y \Rightarrow x' \neq x' \cdot y.$

36. $\forall x, y \in N$, with $y \neq 1, x \neq x \cdot y.$

37. $\forall y, z \in N, 1 \cdot y = 1 \cdot z \Rightarrow y = z.$

38. $\forall x, y, z \in N, (x \cdot y = x \cdot z \Rightarrow y = z) \Rightarrow (x' \cdot y = x' \cdot z \Rightarrow y = z).$

39. $\forall x, y, z \in N, x \cdot y = x \cdot z \Rightarrow y = z.$