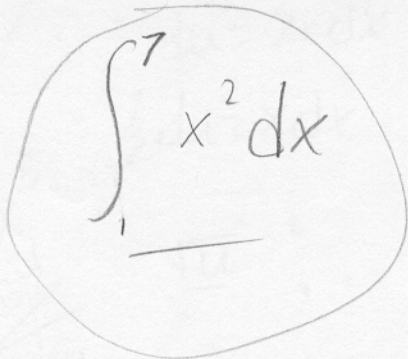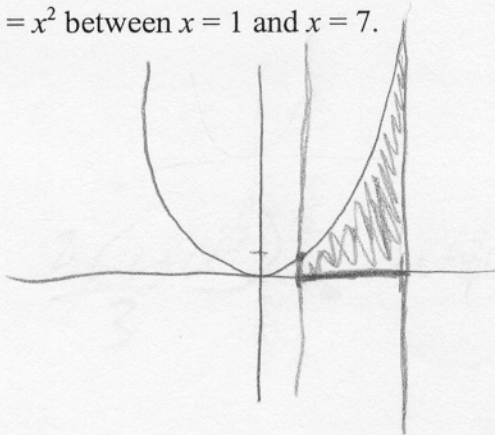Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Write an integral representing the area between $y = 0$ and $y = x^2$ between $x = 1$ and $x = 7$.

$$\int_{1}^{7} x^2 \, dx$$

yep.

2. Write an integral representing the average value of the function $f(x) = \dfrac{\sin x}{x}$ on the interval $[\pi/2, 3\pi/2]$.

Ave value of function $f(x)$ on an interval $[a,b]$ is

$$\frac{1}{b-a} \int_a^b f(x)\,dx$$

Excellent!

$$\frac{1}{\frac{3\pi}{2} - \frac{\pi}{2}} \int_{\pi/2}^{3\pi/2} \frac{\sin x}{x}\,dx = \left(\frac{1}{\pi} \int_{\pi/2}^{3\pi/2} \frac{\sin x}{x}\,dx \right)$$

3. Evaluate the integral $\int \sqrt{1+x^2}\, x^5\, dx$.

$$\int \sqrt{1+x^2} \times x \times x^4\, dx$$

$$u = 1+x^2$$
$$du = 2x\, dx$$
$$\frac{du}{2}\, dx$$

$$x^2 = (u-1)$$
$$x^4 = (u-1)^2$$

$$\frac{1}{2} \int \sqrt{u}\, (u-1)^2\, du$$

$$\int \sqrt{u}\, (u-1)(u-1)$$

$$\frac{1}{2} \int \sqrt{u}\, (u^2 - u - u + 1)\, du$$

$$\frac{1}{2} \int \sqrt{u}\, (u^2 - 2u + 1)\, du$$
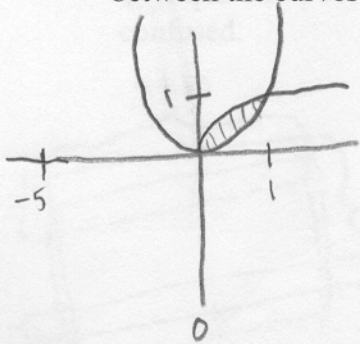
$$\frac{1}{2} \int u^{2.5} - 2u^{1.5} + u^{1/2}\, du$$

$$\frac{1}{2} \left(\frac{u^{3.5}}{3.5} - \frac{2u^{2.5}}{2.5} + \frac{u^{1.5}}{1.5} \right)\Big|$$

Good
Work

$$\frac{1}{2} \left[\frac{2}{7} (1+x^2)^{7/2} - \frac{4}{5} (1+x^2)^{5/2} + \frac{2}{3} (1+x^2)^{3/2} \right] + C$$

4. Write an integral for the volume of the solid generated by rotating the region bounded between the curves $y = x^2$ and $x = y^2$ around the axis $x = -5$.

$y = x^2$
$y = \sqrt{x}$

$$2\pi \int_0^1 (x+5)(\sqrt{x} - x^2)\,dx$$

Great

Shell Method

axis point

Top – Bottom

−5

0

1

10

5. If a spring has a natural length of 30cm, and 5J of work is required to stretch it to a length of 40cm, how much work would be required to stretch it from 30cm to 35cm?

$30cm = .3m$

$$5 = \int_0^{.1} Kx \, dx$$

$$5 = \frac{1}{2}Kx^2 \Big|_0^{.1}$$

$$5 = \frac{.01K}{2} - 0$$

$$1000 = K$$

$$F = Kx$$

$$W = \int_0^{.05} 1000x$$
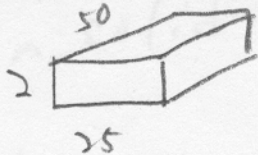
$$= 500x^2 \Big|_0^{.05}$$

$$W = 1.25 \text{ J of work}$$

good

6. An Olympic-size swimming pool is 2 meters deep, 25 meters wide, and 50 meters long. Set up an integral for the amount of work required to pump all of the water in a (completely full) Olympic-size pool out over the edge of the pool, while Michael Phelps sits nearby looking confused.

AREA of SLICE = 50×25

VOLUME of SLICE = $50 \times 25 \times \Delta x$

FORCE EXERTED by SLICE = $(9.8)(1000)(50)(25) \Delta x$

$$W = \int_0^2 (9.8)(1000)(50)(25)(x)\,dx$$

Good

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, this solid revolution stuff is way too hard for me. I heard these foreign guys in my class talk about it, though, and they was sayin' there was this way where, like, you just take the area of the stuff, like before it gets rotated, right? And you just take that times two pi and it gives you the volume. It figures they wouldn't tell us anything that easy, 'cause then everyone would pass the class, but now that I know I'm gonna blow through our exam tomorrow!"

Explain clearly to Biff why his proposed approach should or should not be depended on.

Look Biff, aren't you suspicious that's too easy to really handle all the different solids of revolution you've seen?
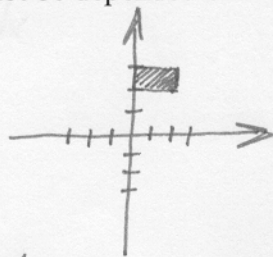
Think about the rectangle at right. It has area 2.

Your way says if it's rotated you get volume $2\pi \cdot 2 = 4\pi$.

But you can tell it should work out different going around the x- or y-axis, right? Around the y-axis it makes a cylinder with height 1 and radius 2, for volume 4π, but around the x-axis it's the difference between a cylinder with radius 3 and a cylinder with radius 2, both with height 2, for volume $18\pi - 8\pi = 10\pi$.

So maybe you misheard, or maybe the other students were just wrong, but it definitely isn't always so simple.

8. Consider the regions bounded between curves of the form $y = x - x^a$ and the x-axis for integer values of a which are greater than or equal to 2. If such a region is rotated around the y-axis, what is the volume of the region created?

$$X - X^2$$
$$X - X^3$$
$$X - X^4$$

$$\frac{2\pi \int_0^1 x\left[x - x^a\right] dx}{}$$

$$2\pi \int_0^1 \left(x^2 - x^{a+1}\right) dx$$

$$2\pi \left[\frac{x^3}{3} - \frac{x^{a+2}}{a+2}\right]_0^1$$

$$2\pi \left[\frac{1}{3} - \frac{1}{a+2}\right]$$

Great!

9. The city of Ringville has a population density of 100,000 people per square mile at its center, which decreases steadily to 50,000 people per square mile at a distance of 5 miles from the center of the city. What can you say about the total population living within 5 miles of the center of Ringville?

x	P
0	100 000
5	50 000

Just to gauge answer...

High $\frac{2}{3} = 100000 \cdot \pi \cdot 5^2 = 2500000\pi$ people

Low $= 50000 \cdot \pi \cdot 5^2 = 1250000\pi$ people

Sample Ring -
Area $\approx 2\pi \cdot$ radius,

$$m = \frac{50000 - 100000}{5 - 0} = \frac{-50000}{5} = -10000$$

$$P - 100000 = -10000(x - 0)$$

$$P = -10000x + 100000$$
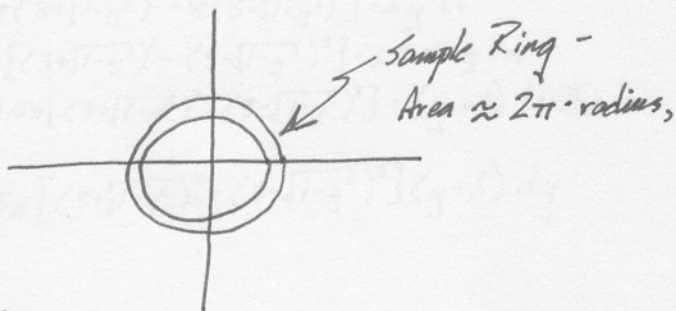
$$\text{Total} = \int_0^5 \overbrace{2\pi \cdot x}^{\text{Area}} \cdot \overbrace{(100000 - 10000x)}^{\text{Density}} dx$$

$$= \int_0^5 2\pi (100000x - 10000x^2) dx$$

$$= 2\pi \left[50000x^2 - \frac{10000}{3}x^3 \right]_0^5$$

$$= 2\pi \cdot \left[\left(1250000 - \frac{1250000}{3} \right) - (0 - 0) \right]$$

$$= \frac{4\pi}{3} \cdot 1250000$$

$$\approx 5,235,988 \text{ people}$$

And since that's between our initial over- and under-estimates, it seems good!

10. A torus is the solid created when a circle is rotated around an axis outside the circle (so the shape of a donut, or an inner tube). Suppose a torus is formed by rotating the circle $(x-3)^2 + y^2 = 1$ around the y-axis (where the scale is in feet), and that the circle is filled with water (which weighs 62.5 lbs per ft^3). Write an integral for the amount of work required to pump all of the water out of the top of the torus. [If you can't manage the full integral, at least give an intelligent approximation.]

Seen from side:
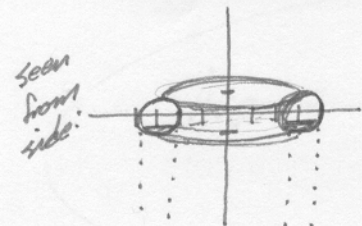
Top view:
Region between two circles
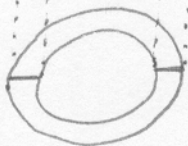
$$(x-3)^2 + y^2 = 1$$
$$(x-3)^2 = 1 - y^2$$
$$x - 3 = \pm\sqrt{1-y^2}$$
$$x = 3 \pm \sqrt{1-y^2}$$

Area of a slice $= \pi R_1^2 - \pi R_2^2$ ft^2

$$= \pi\left(3+\sqrt{1-y^2}\right)^2 - \pi\left(3-\sqrt{1-y^2}\right)^2 \text{ ft}^2$$

Volume " " " $= \left[\pi\left(3+\sqrt{1-y^2}\right)^2 - \pi\left(3-\sqrt{1-y^2}\right)^2\right] \cdot \Delta y \text{ ft}^3$

Force " " " $= \pi\left[\left(3+\sqrt{1-y^2}\right)^2 - \left(3-\sqrt{1-y^2}\right)^2\right] \cdot 62.5 \Delta y \text{ lbs}$

Work " " " $= 62.5\pi\left[\left(3+\sqrt{1-y^2}\right)^2 - \left(3-\sqrt{1-y^2}\right)^2\right] \cdot (y+1) \text{ ft}\cdot\text{lbs}$

Total Work $= \int_{-1}^{1} 62.5\pi\left[\left(3+\sqrt{1-y^2}\right)^2 - \left(3-\sqrt{1-y^2}\right)^2\right](y+1)\, dy$