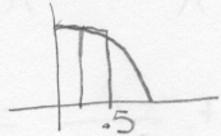
Exam 2 Calc 2 3/6/2009

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Find L_2 for $\int_0^{0.5} e^{-x^2}\,dx$.



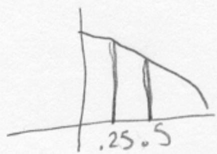
$\Delta x = \frac{.5-0}{2}$

$\frac{1}{4}\left(e^{-0^2} + e^{-.25^2}\right)$

$1/4\,(1 + .939) \approx \boxed{.48475}$

Great!

b) Find M_2 for $\int_0^{0.5} e^{-x^2}\,dx$.



$\Delta x = \frac{.5}{2}$

$\frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(3/8\right)\right)$

$1/4\,(.984 + .869) \approx \boxed{.46325}$

2. Set up an integral for the area of the surface obtained by rotating the portion of $y = x^3$ on the interval [0,2] about the x- axis.

$f'(x) = 3x^2$

Area of surafe $= \int_a^b 2\pi\, f(x)\sqrt{1+[f'(x)]^2}\, dx$

$= \int_0^2 2\pi\, x^3 \sqrt{1+(3x^2)^2}\, dx$.

Good

3. Write out the form of the partial fraction decomposition of the function

$$\frac{x^3-x+1}{x(x-2)\underbrace{(x^2+x+1)}_{\text{non repeated}}(x^2+1)^3}.$$

irreducible quadratic repeated

The degree of denom. is greater than numerator

Excellent

$$\frac{A}{x}+\frac{B}{x-2}+\frac{Cx+D}{x^2+x+1}+\frac{Ex+F}{x^2+1}+\frac{Gx+H}{(x^2+1)^2}+\frac{Jx+K}{(x^2+1)^3}$$

4. Evaluate $\int_1^{\infty} \frac{1}{x^3}\, dx$.

$$\int_1^{\infty} \frac{1}{x^3}\, dx = \underline{\lim_{b\to\infty} \int_1^b x^{-3}\, dx} = \underline{\lim_{b\to\infty} \left[\frac{x^{-2}}{-2} \right]_1^b}$$

$$\underline{\lim_{b\to\infty} \left[\frac{-1}{2x} \right]_1^b}$$

Great!

$$\underline{\lim_{b\to\infty} \left[\left(\underbrace{\frac{-1}{2b}}_{0} \right) - \left(\frac{-1}{2} \right) \right]} = \underline{\frac{1}{2}}$$

5. Show that $\int u^4 \sqrt{a^2 - u^2}\, du$ can be transformed by an appropriate substitution into $a^6 \int \sin^4 \theta \cos^2 \theta \, d\theta$.

suppose $\underline{u = a\sin\theta}$ $\underline{du = a\cos\theta\, d\theta}$ $1 - \sin^2\theta = \cos^2\theta$

then $\sqrt{a^2 - u^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = \underline{a\cos\theta}$

$$\int u^4 \sqrt{a^2 - u^2}\, du$$

$$= \int \underline{(a\sin\theta)^4} \cdot \underline{a\cos\theta} \cdot \underline{a\cos\theta\, d\theta}$$

$$= a^6 \int \sin^4\theta \cos^2\theta \, d\theta$$

Great

6. Find the length of the curve $y = \ln(\sec x)$ on the interval $[0, \pi/4]$.

$y = \ln(\sec x)$

$y' = \tan x$

$$C = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$C = \int_0^{\pi/4} \sqrt{1 + (\tan^2 x)}\, dx$$

ldne 14

$$= \int_0^{\pi/4} \sec x\, dx$$

$$= \Big[\ln|\sec x + \tan x|\Big]_0^{\pi/4}$$

$$= \Big[\ln|\sqrt{2} + 1| - \ln|1 + 0|\Big]$$

Excellent

$$= \ln|\sqrt{2} + 1|$$

7. Derive line 84 on the table of integrals.

$$\int x^n \sin x \, \partial x = -x^n \cos x + n \int x^{n-1} \cos x \, \partial x$$

let $u = x^n$ $\quad v = -\cos x \, \partial x$

$du = n x^{n-1} dx \quad dv = \sin x \, dx$

by parts

$$-x^n \cos x \, \partial x - \int (-\cos x)(n x^{n-1}) \, \partial x$$

Great

$$-x^n \cos x \, \partial x + n \int (x^{n-1}) \cos x \, \partial x$$

8. You have been tasked with writing a section for the forthcoming book *Incredibly Rarely Used Techniques in Calculus*. The section is to cover integrating combinations of csc x and cot x. Explain, in terms a typical calculus student can follow, a basic procedure for integrating products of powers of these functions.

Before getting into the procedure we know,

$$1+\cot^2 x = \csc^2 x \quad \& \quad f' \cot x = -\csc^2 x$$

$$f' \csc x = -\csc x \cot x.$$

Now, in

If ^in the combination ~~has a~~ the power of csc x is even $(n=2k, k \geq 2)$, we need to take a factor of $\csc^2 x$. Then using the above stated identity, $1+\cot^2 x = \csc^2 x$, we should change the reamaning terms into cot x.

a, if $\int \cot^m x \csc^{2k} x\,dx$ ~~then~~ $\int \cot^m x \left(1+\cot^2 x\right)^{k-1} \cdot \csc^2 x\,dx$

then we simply take $u = \cot x$ and our intregation is so much is easier. The only thing to remember is to add a minus sign.

where as

If the combination has an odd power of cot $(m=2k+1)$

- We should take a factor of csc x and cot x and then use $\cot^2 x = \csc^2 x - 1$ to change the remaning terms into csc x.

a, if $\int \cot^{2k+1} \cdot \cos^n x\,dx = \int (\cot^2 x - 1)^k \csc^{n-1} \cdot \csc x \cdot \cot x\,dx$

then we simply take $u = \cot x$ and our work is almost done. The rest is a simple integration with u subtitution.

Excellent!

9. Suppose that $p(t) = \begin{cases} \frac{1}{4}e^{-t/4} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$ is a p.d.f. representing a probability that a computer armoire purchased from Home Design Solutions breaks within t weeks of purchase.

a) Find the median of this p.d.f.

The median is the value such that half the population lies below it, so:

$$\frac{1}{2} = \int_{-\infty}^{b} p(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{b} \frac{1}{4}e^{-t/4}\,dt = 0 + \int_{0}^{b} \frac{1}{4}e^{-t/4}\,dt$$

which, by a little u-substitution,

$$= -e^{-t/4}\Big|_0^b = -e^{-b/4} - -e^{0} = 1 - \frac{1}{e^{b/4}}$$

But all that was supposed to total 1/2, so

$$\frac{1}{2} = 1 - \frac{1}{e^{b/4}}$$

$$-\frac{1}{2} = -\frac{1}{e^{b/4}}$$

$$e^{b/4} = 2$$

$$b/4 = \ln 2$$

$$b = 4 \cdot \ln 2$$

b) A cumulative distribution function $c(t)$ associated with a given p.d.f. $p(t)$ is a function which, for each value of t, gives the proportion of the sample less than t. Find $c(t)$ for Home Design Solutions computer armoires.

Well, the proportion of the sample less than t is exactly what $\int_{-\infty}^{t} p(t)\,dt$ tells us, so

$$c(t) = \int_{-\infty}^{t} p(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_{0}^{t} \frac{1}{4}e^{-t/4}\,dt \quad (\text{as long as } t \geq 0)$$

$$= -e^{-t/4}\Big|_0^t$$

$$= -e^{-t/4} - -e^{0}$$

$$= 1 - e^{-t/4}$$

10. Consider the trapezoidal region bounded by $x = 0$, $y = 0$, $x = 1$, and a line with y-intercept 1 and slope m.

a) If $m = 1$, find the x coordinate of the center of mass of the trapezoidal region.

b) For other positive constant values of m, how large can the x coordinate of the center of mass of the region get?

a) A line with y-intercept 1 and slope 1 has equation $y = x+1$, so

$$\bar{x} = \frac{\int_0^1 x(x+1)\,dx}{\int_0^1 (x+1)\,dx} = \frac{\int_0^1 (x^2+x)\,dx}{\int_0^1 (x+1)\,dx} = \frac{\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1}{\left[\frac{x^2}{2}+x\right]_0^1} = \frac{\frac{1}{3}+\frac{1}{2}}{\frac{1}{2}+1} = \frac{\frac{5}{6}}{\frac{3}{2}} = \frac{5}{9}$$

b) Now with y-intercept 1 and slope m we have $y = mx+1$, so

$$\bar{x} = \frac{\int_0^1 x(mx+1)\,dx}{\int_0^1 (mx+1)\,dx} = \frac{\int_0^1 (mx^2+x)\,dx}{\int_0^1 (mx+1)\,dx} = \frac{\left[\frac{mx^3}{3}+\frac{x^2}{2}\right]_0^1}{\left[\frac{mx^2}{2}+x\right]_0^1} = \frac{\frac{m}{3}+\frac{1}{2}}{\frac{m}{2}+1} = \frac{2m+3}{3m+6}$$

Then as m grows bigger and bigger, we're looking at

$$\lim_{m\to\infty} \frac{2m+3}{3m+6} = \frac{2}{3}$$

So $\bar{x}$ grows closer and closer to, but without reaching, $\frac{2}{3}$.