Examlet 1 Foundations of Advanced Math 2/6/09

1. a) Determine whether the propositionals $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$) are equivalent.

P	Q	$P \Rightarrow Q$ *
T	T	T
T	F	F
F	T	T
F	F	T

$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$ *
F	F	T
F	T	F
T	F	T
T	T	T

The columns marked * are equivalent, so the two statements are equivalent. □

Good

b) Determine whether the propositionals $(P \wedge Q) \Rightarrow R$ and $(P \Rightarrow R) \vee (Q \Rightarrow R)$ are equivalent.

P	Q	R	$P \wedge Q$	$(P \wedge Q) \Rightarrow R$ *	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$ *
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	T	F	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

The columns marked * are equivalent, so the two statements are equivalent. □

Great

2. If n divides a and n divides b, then n divides $a + b$.

If n divides a, then by our definition of divides

$a = nx \quad x \in \mathbb{Z}$

Similarly, it can be shown that

$b = ny \quad y \in \mathbb{Z}$

Then:

$a+b = nx + ny = n(x+y) = nz \quad z = x+y$

By the closure of integers, z is an integer. Therefore, a+b is the product of an integer n and an integer z, $\therefore$ n divides a+b. $\square$

Well done.

3. $\sqrt[3]{2}$ is irrational.

S'pose not. S'pose $\sqrt[3]{2}$ is rational. Then

$\sqrt[3]{2} = \frac{a}{b}$ for some a, b $\in \mathbb{Z}$ and in lowest terms.

$$2 = \frac{a^3}{b^3}$$

$$2b^3 = a^3$$

So we see that a^3 is an even number. Since any odd cubed is an odd number, we know that a must also be even. Then

$a = 2x$ where $x \in \mathbb{Z}$

$$2b^3 = (2x)^3$$

$$b^3 = 4x^3$$

$$b^3 = 2(2x^3)$$

Excellent

So, similarily we see that b^3 is also even, and that b must be even. However, this means that a and b share a common term, namely 2. This contradicts a and b being in lowest terms. Our supposition must be false. $\sqrt[3]{2}$ must be irrational. $\square$

4. Prove that $\forall n \in \mathbb{N}, \sum_{r=1}^{n}(2r-1) = n^2$. define $P(n)$ to be this statement

We'll induct.

Base case:

$n=1$ $\quad 2(1)-1=1 \quad (1)^2=1 \quad 1=1$ The proposition is true at $n=1$.

Inductive case:

We must show that $P(n) \Rightarrow P(n+1)$

$\sum_{r=1}^{n+1}(2r-1)$ can be rewritten as $\sum_{r=1}^{n}(2r-1) + 2(n+1)-1$

By $P(n)$, this in turn can be written as $n^2+2(n+1)-1$

$n^2+2(n+1)-1 = n^2+2n+2-1 = n^2+2n+1 = (n+1)^2$

Therefore, $\sum_{r=1}^{n+1}(2r-1) = (n+1)^2$, and the inductive case is true.

Since the prop. holds for $n=1$, and if it holds at n it holds at $n+1$, the original proposition is proved by induction.

Well done!

5. We say that an integer *m* is **congruent to 0 modulo 5** iff $m = 5n$ for some integer *n*.
We say that an integer *m* is **congruent to 1 modulo 5** iff $m = 5n + 1$ for some integer *n*.
We say that an integer *m* is **congruent to 2 modulo 5** iff $m = 5n + 2$ for some integer *n*.
We say that an integer *m* is **congruent to 3 modulo 5** iff $m = 5n + 3$ for some integer *n*.
We say that an integer *m* is **congruent to 4 modulo 5** iff $m = 5n + 4$ for some integer *n*.

a) If *a* is congruent to 1 modulo 5, then a^2 is congruent to 1 modulo 5.

If a is congruent to 1 modulo 5, it can be written as $5n+1$ for some integer n. This gives us:

$$\begin{aligned} a^2 &= (5n+1)^2 \\ &= 25n^2 + 10n + 1 \\ &= 5(5n^2+2n)+1 \end{aligned}$$

The expression inside the parentheses is an integer due to closure, so it could be written as $5(m)+1$ for some integer $m = 5n^2+2n$.
Thus, $a^2 = 5m+1$, so our proposition is true. □

Good

b) If *a* is an integer for which a^2 is congruent to 1 modulo 5, then *a* is congruent to 1 modulo 5.

Suppose a is not congruent to 1 modulo 5. As a must be an integer, it must therefore be congruent to 0, 2, 3 or 4 modulo 5.
Let's examine these cases.

a	a^2		a is...	a^2 is...
$5n$	$25n^2$	$= 5(5n^2)$	congruent to 0 mod 5	congruent to 0 mod 5
$5n+2$	$25n^2+20n+4$	$= 5(5n^2+4n)+4$	congruent to 2 mod 5	congruent to 4 mod 5
$5n+3$	$25n^2+30n+9$	$= 5(5n^2+6n+1)+4$	congruent to 3 mod 5	congruent to 4 mod 5
$5n+4$	$25n^2+40n+16$	$= 5(5n^2+8n+3)+1$	congruent to 4 mod 5	congruent to 1 mod 5

We can see from the above table that a^2 is also congruent to 1 modulo 5 if a is congruent to 4 modulo 5, so our proposition cannot always hold true. □

Yes!