**Examlet 3 Foundations of Advanced Math 3/27/09**

1. Let $a$ and $b$ be real numbers with $a < b$.

a) Give an example of a bijection from [0,1] to [$a$,$b$].

$f(x) = (b-a)x + a$ is a bijection, since it takes 0 to $a$ and 1 to $b$ and is increasing and continuous.

I want a line through $(0,a)$ and $(1,b)$, so

$$m = \frac{b-a}{1-0} = b-a$$

Then

$$y-(a) = (b-a)(x-(0))$$

$$y = (b-a)x + a$$

b) Give an example of a function from [0,1] to [$a$,$b$] which is not a bijection.

$g(x) = a$ is not one-to-one, so it's not a bijection.

2. Let $f: A \to B$ and $g: B \to C$ be injective functions. Show that $g \circ f$ is injective.

Let $g \circ f = h(x)$

Let $a_1$ and $a_2$ be elements of A

Let $h(a_1) = h(a_2)$, then $g(f(a_1)) = g(f(a_2))$ and since g is injective, $f(a_1) = f(a_2)$ and since f is injective, $a_1 = a_2$.

For $h(a_1) = h(a_2)$, it must be true that $a_1 = a_2$, so we know $h(x)$ or $g \circ f$ is injective because each output corresponds to a unique input. □

Great.

3. Let $f$:$A \to B$. What is $f \circ f^{-1}$? Support your answer.

$f: A \to B$ $\quad$ $f(a) = b \to f^{-1}(b) = a$

$f^{-1}: B \to A$

$f \circ f^{-1}$

$= f(f^{-1}(b))$ ← Because $B$ is the domain for $f^{-1}$

Identity Function

$\boxed{I_B(b) = b \quad \forall b \in B}$

$= f(a)$

$= b$ $\quad$ $f(f^{-1}(b)) = b$

$f \circ f^{-1}$ inputs a value $b$, and also outputs that value $b$. So essentially $f \circ f^{-1}$ is equal to the identity function because they share a domain, codomain, and

$f(f^{-1}(b)) = I_B(b)$. □

Excellent

4. a) Show that the set of even natural numbers is denumerable.

Let $f(x) = 2x$ be a function from the naturals to the even naturals. Then $f$ is onto since every even natural is of the form $2n$ for some $n \in \mathbb{N}$, and $f$ is one-to-one since it's increasing, so $f$ is a bijection as desired. $\square$

b) Show that if *A* and *B* are two denumerable sets, then there exists a bijection from *A* to *B*.

Since $A$ is denumerable we know there exists $f: \mathbb{N} \to A$ a bijection, and since $B$ is denumerable we know there also exists $g: \mathbb{N} \to B$ a bijection. Then since $f$ is a bijection, $f^{-1}: A \to \mathbb{N}$ is also a bijection, and since the composition of bijections is a bijection we have $g \circ f^{-1}: A \to B$ as the desired bijection. $\square$

5. Suppose that $f: A \to B$ is a bijection. Show that $f$ is invertible.

Well, for any $b \in B$, since $f$ is surjective we know there exists $a \in A$ such that $f(a) = b$. We also know, since $f$ is injective, that there is only one such $a$ for a given $b$. Then we define $f^{-1}(b) = a$, and note that $f(a) = b \Rightarrow f^{-1}(b) = a$ and $f^{-1}(b) = a \Rightarrow f(a) = b$, satisfying the definition of an inverse function as desired. $\square$