

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Let  $F(x) = \int_0^x \cos(t^2) dt$ . What is  $F'(x)$ ?

By the Fundamental Theorem of Calculus.

$F'(x) = \cos(x^2)$

Yes!

2. Write an integral representing the average value of the function  $f(x) = \frac{\sin x}{x}$  on the interval  $[\pi/2, 3\pi/2]$ .

$$\text{A.V.} = \frac{1}{b-a} \int_a^b f(x) dx$$

$$\frac{1}{\frac{3\pi}{2} - \frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left( \frac{\sin x}{x} \right) dx$$

$$\text{A.V.} = \frac{1}{\pi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left( \frac{\sin x}{x} \right) dx$$

Great!

3. Integrate  $\int x \cos(x^2) dx$ .

$$\underline{u = x^2}$$

$$\frac{du}{dx} = 2x$$

$$\underline{\frac{du}{2x} = dx}$$

$$\int x \cos(u) \frac{du}{2x}$$

$$= \int \frac{1}{2} \cos(u) du$$

$$= \left(\frac{1}{2}\right) \sin(u) = \left(\frac{1}{2}\right) \sin(x^2) + C$$

check! derivative of  $\frac{1}{2} \sin x^2 + C$

$$\frac{1}{2} (\cos x^2) (2x) + 0 = x \cos(x^2) \checkmark$$

Wonderful!

4. If a spring has a natural length of 10cm, and 50N of force is required to hold it stretched to 12cm, how much work would be required to stretch it from 10cm to 12cm?

$$12 \text{ cm} - 10 \text{ cm} = 2 \text{ cm} = 0.02 \text{ m} \quad F = kx$$

$$50 \text{ N} = k(0.02 \text{ m})$$

$$k = 2500 \text{ N/m}$$

$$\int_0^{0.02} 2500 x \, dx$$

$$\left[ 1250 x^2 \right]_0^{0.02}$$

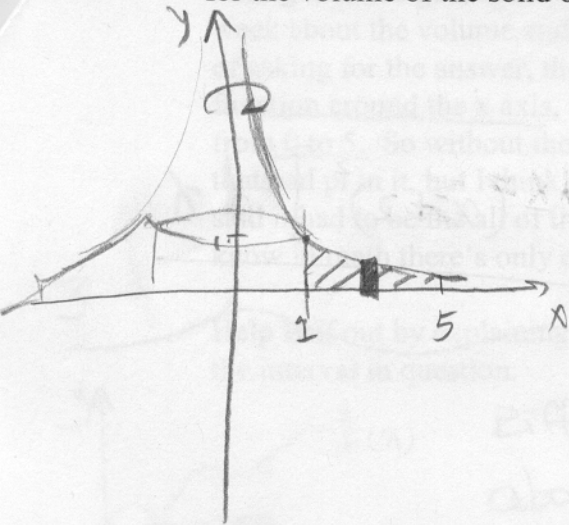
$$w = 0.5 \text{ Joules}$$

$$F = 40 \text{ N}$$

$$x = 0.02 \text{ m}$$

Great

5. Consider the region bounded between  $y = 1/x$ , the  $x$ -axis,  $x = 1$ , and  $x = 5$ . Write an integral for the volume of the solid obtained by rotating this region around the  $y$ -axis.



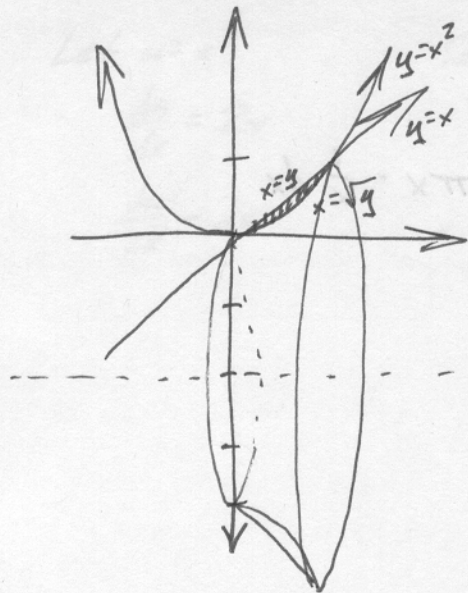
Shell:

$$\int_1^5 (2\pi x) \left(\frac{1}{x}\right) dx$$

$$= 2\pi \int_1^5 1 dx$$

Good

6. Consider the region bounded between  $y = x$ , and  $y = x^2$ . Write an integral for the volume of the solid obtained by rotating this region around the axis  $y = -2$ .



Washers

$$\text{Volume} = \pi \int_0^1 \left[ \overbrace{(x+2)^2}^{\text{Outer Radius}} - \overbrace{(x^2+2)^2}^{\text{Inner Radius}} \right] dx$$

Shells

$$\text{Volume} = \int_0^1 2\pi \overbrace{(y+2)}^{\text{Radius}} \overbrace{(\sqrt{y}-y)}^{\text{Height}} dy$$

7. Biff is a Calc 2 student at Enormous State University. Biff says "Dang, this Calc stuff is killing me. It's getting all, like, theoretical. There was this one question on our exam last week about the volume stuff, which is pretty much the toughest stuff ever, right? So instead of asking for the answer, though, it was like, what *could* the volume be if you rotate some function around the x axis, but they don't say what the function is, just that it's on the interval from 0 to 5. So without the function, I had no clue what to do, so I just picked the answer that had pi in it, but I think maybe I should have said none of the above. This girl in my class said it had to be the all of the above were possible, but she's obviously stupid because you know in math there's only one right answer."

Help Biff out by explaining clearly what possible volumes solids of revolution might have on the interval in question.

Biff, I'm pretty sure you missed that one, and the girl was right. It's true each solid of revolution just has one volume, but there are lots of solids of revolution. Even if you just think about really simple solids, like cylinders that come from rotating constant functions, you can get any positive volume (or even zero) just by picking the right radius. Like, if you want a volume of 7, then a cylinder with radius  $r = \sqrt{\frac{7}{5\pi}}$  will give  $V = \pi \left( \sqrt{\frac{7}{5\pi}} \right)^2 (5) = 7$ .

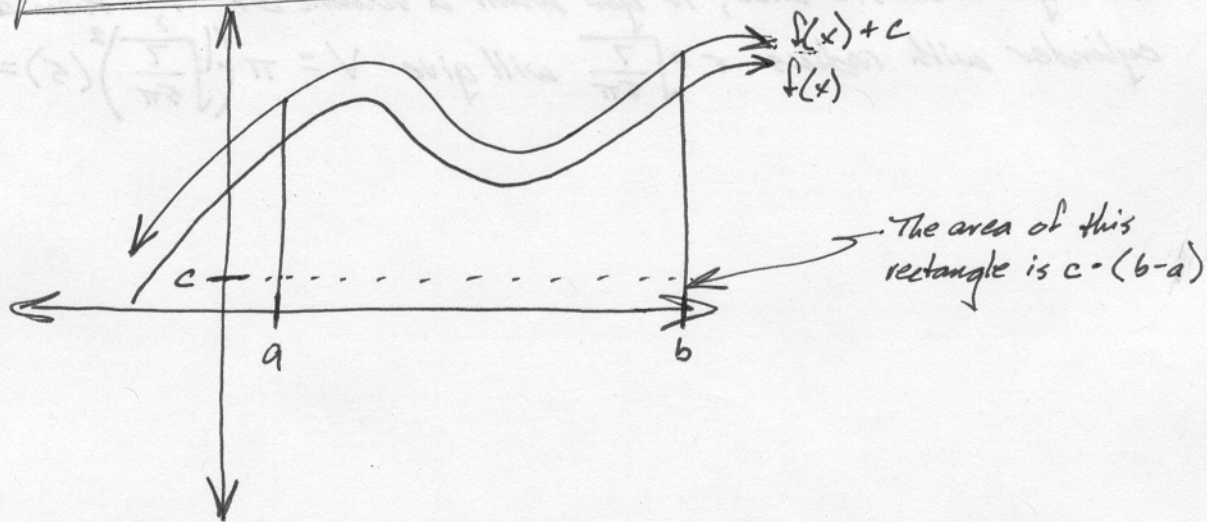
8. a) Find a formula relating  $\int_a^b [f(x) + c] dx$  and  $\int_a^b f(x) dx$ .

b) Draw a picture to illustrate the relation you found in part a, and describe why it makes geometric sense.

Well,  $\int_a^b [f(x) + c] dx = \int_a^b f(x) dx + \int_a^b c dx$ , since integrals distribute over sums. And

$$\int_a^b c dx = cx \Big|_a^b = c \cdot b - c \cdot a, \text{ or } c(b-a).$$

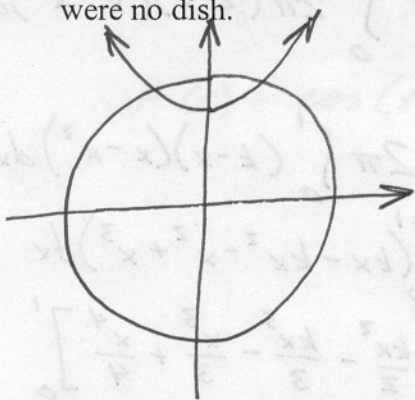
So  $\int_a^b [f(x) + c] dx = \int_a^b f(x) dx + c(b-a)$



This relation makes sense because it just says shifting the graph up vertically by  $c$  units adds  $c$  times the interval length to the integral. It's like you add that strip across the bottom of the region, so that's exactly how much the area changes.



9. In the movie *Star Wars VI, The Return of the Jedi*, the new Death Star is a sphere 160 kilometers in diameter, so it could be obtained as a solid of revolution by rotating part of  $x^2 + y^2 = 80^2$  around the  $y$ -axis. The reflector dish for the planet-destroying laser projector is (if we arrange the  $y$ -axis to run through its center) the paraboloid obtained by rotating  $y = x^2/100 + 60$ . Write an integral for the volume of the reflector dish, that is, the volume of the region between the bottom of the dish and what would have been the surface of the sphere if there were no dish.



Where does  $x^2 + y^2 = 80^2$  intersect  $y = \frac{x^2}{100} + 60$ ?

$$\Downarrow$$

$$100y = x^2 + 6000$$

$$x^2 = 100y - 6000$$

$$(100y - 6000) + y^2 = 80^2$$

$$y^2 + 100y - 6000 - 6400 = 0$$

$$y^2 + 100y - 12400 = 0$$

Quadratic Formula:

$$y = \frac{-(-100) \pm \sqrt{(-100)^2 - 4(1)(-12400)}}{2(1)}$$

$$= \frac{-100 \pm \sqrt{10000 + 49600}}{2}$$

$$= \frac{-50 \pm \sqrt{59600}}{2}$$

$$= -50 \pm 10\sqrt{149}$$

Pick positive root based on picture

And the corresponding  $x$  value is

$$x^2 = 100(-50 + 10\sqrt{149}) - 6000$$

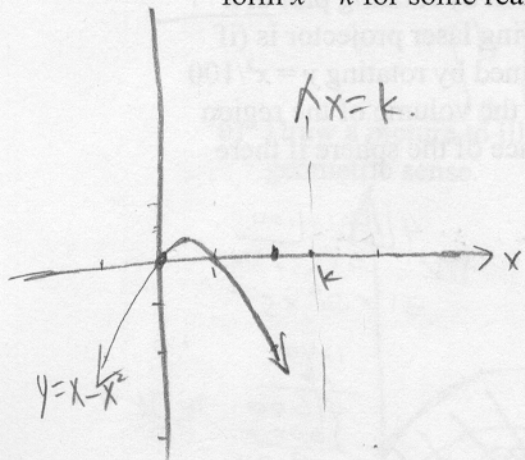
$$x^2 = 1000\sqrt{149} - 11000$$

$$x = \pm \sqrt{1000\sqrt{149} - 11000}$$

Shells

$$\text{Volume} = \int_0^{\sqrt{1000\sqrt{149} - 11000}} 2\pi x \left( \sqrt{80^2 - x^2} - \left( \frac{x^2}{100} + 60 \right) \right) dx$$

10. The region bounded between  $y = x - x^2$  and the  $x$ -axis is to be rotated around an axis of the form  $x = k$  for some real value of  $k$ . Find a value for  $k$  that produces a solid with volume  $6\pi$ .



$x=k$

$k > 1$  or  $k < 0$   
Shell

$$2\pi \int_0^1 (k-x)(x-x^2) dx = 6\pi$$

$$\int_0^1 (k-x)(x-x^2) dx = 3$$

$$= \int_0^1 (kx - kx^2 - x^2 + x^3) dx$$

$$= \left[ \frac{kx^2}{2} - \frac{kx^3}{3} - \frac{x^3}{3} + \frac{x^4}{4} \right]_0^1$$

$$= \frac{k}{2} - \frac{k}{3} - \frac{1}{3} + \frac{1}{4} = 3$$

$$\frac{k}{6} = 3 + \frac{1}{3} - \frac{1}{4} = 3 + \frac{4-3}{12} = \frac{36+1}{12} = \frac{37}{12}$$

$$k = \frac{37}{2} = 18\frac{1}{2}$$

$18\frac{1}{2} > 1$  it works.

$\therefore$  when  $k = \frac{37}{2}$ , the solid's volume is  $6\pi$ .

Excellent!