Exam 3a **Calc 2** **4/1/2011**

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Write the first three terms in the sequence $\left\{ \frac{n}{n+2} \right\}$.

$$a_1 = \frac{(1)}{(1)+2} = \underline{\frac{1}{3}}$$

$$a_2 = \frac{(2)}{(2)+2} = \frac{2}{4} = \underline{\frac{1}{2}}$$

$$a_3 = \frac{(3)}{(3)+2} = \underline{\frac{3}{5}}$$

b) Write the first three partial sums of the series $\sum_{n=1}^{\infty} \frac{n}{n+2}$.

Good

$$S_1 = \underline{\frac{1}{3}}$$

$$S_2 = \frac{1}{3} + \frac{1}{2} = \underline{\frac{5}{6}}$$

$$S_3 = \frac{5}{6} + \frac{3}{5} = \underline{\frac{43}{30}}$$

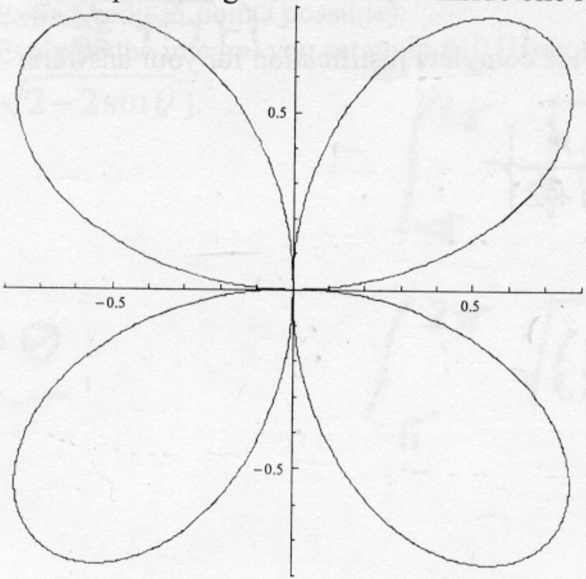
2. Find the sum of the series $\frac{3}{4}-\frac{1}{2}+\frac{1}{3}-\frac{2}{9}+\frac{4}{27}-\ldots$.

Geometric Series with $a = 3/4$

$r = -2/3$ (Note $|-2/3| < 1$ so it's okay)

$$S = \frac{(3/4)}{1-(-2/3)} = \frac{3/4}{5/3} = \frac{9}{20}$$

3. Set up an integral for the area inside one loop of the curve $r = \sin 2\theta$.



$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} (r^2)\, d\theta$$

$$\frac{1}{2}\int_{0}^{\pi/2} (\sin 2\theta)^2\, d\theta$$

Good

$r = \sin 2\theta$

$0 = \sin 2\theta$

$\theta = \pi/2$

4. Find a general solution to the differential equation $\frac{dT}{dt} = 0.2(70 - T)$.

$$\frac{dT}{dt} = 0.2(70-T)$$

$u = 70 - T$

$du = -1\,dT$

$$\int \frac{dT}{70-T} = \int 0.2\,dt$$

$$-\int \frac{du}{u} = \int 0.2\,dt$$

$$-\ln|70-T| = 0.2t + C$$

$$|70-T| = e^{-0.2t + C}$$

$$70 - T = A e^{-0.2t}$$

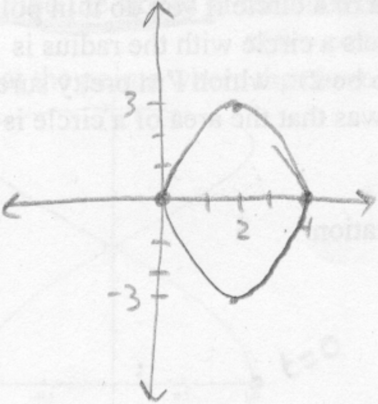
$$70 - T = Ae^{-0.2t}$$

$$-T = A e^{-0.2t} - 70$$

$$T = A e^{-0.2t} + 70$$

Good.

5. Find an equation for the ellipse with vertices at (0,0), (4,0), (2,3), and (2,–3).



$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$\frac{(x-2)^2}{4} + \frac{y^2}{9} = 1$$

Great

6. A cup of coffee starts at 200° F and cools according to the differential equation

$\frac{dT}{dt} = 0.05(70 - T)$. Using $\Delta t = 5$ minutes, approximate the temperature of the coffee after 10 minutes.

t	T	$\frac{dT}{dt}$	Δt
0	200	-6.5	-32.5
5	167.5	-4.875	-24.375
10	143.125		

$\boxed{143.125° F}$

Example of how to get #s

$\frac{dT}{dt} = 0.05(70 - 200)$

$= -6.5$

$\times \Delta t \, (5) = -32.5$

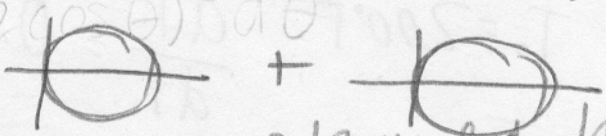
$200 - 32.5 = 167.5$, a.k.a. temp at $t = 5$

Great.

7. Biff is a Calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. There was this question on our test about the area of a circle if you do it in polar, right? And it was like $r = 2\cos\theta$, which I graphed it and it gets a circle with the radius is one, right? But so I did it on my calculator and got the area to be 2π, which I'm pretty sure is wrong. I mean, about the only thing I learned in high school was that the area of a circle is πr^2, and now that's screwed up too!"

Explain clearly to Biff what likely went wrong with his calculation.

Well Biff, when finding the area of shapes in polar coordinates the trickiest part is finding the limits of integration! Many "even" functions will trace the same shape twice, which would give you twice the area you need. If you take the area of the circle from $[0, 2\pi]$ you're esentially adding



because the limits you should have used for integration were $[0, \pi]$. This would only give you one circle!

$$A = \frac{1}{2}\int_{\alpha}^{\beta} (2\cos\theta)^2 \, d\theta$$

Excellent!

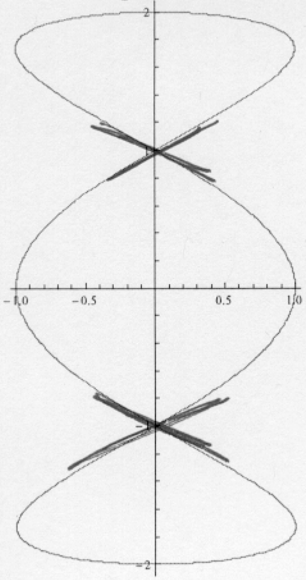
$$A = \frac{1}{2}\int_{0}^{2\pi} (2\cos\theta)^2 \, d\theta = 2\pi$$

$$A = \frac{1}{2}\int_{0}^{\pi} (2\cos\theta)^2 \, d\theta = \pi$$

8. Find the slopes of the lines tangent to the curve with parametric equations

$$x(t) = \cos 3t \qquad y(t) = 2\sin t$$

at the points where the curve crosses itself.



Those are where $x = 0$, so

$$0 = \cos 3t$$

$$\frac{\pi}{2} = 3t$$

$t = \frac{\pi}{6}$ is the first such crossing.

So at the first crossing the slope is

$$\left.\frac{dy}{dx}\right|_{t=\pi/6} = \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi/6}$$

$$= \left.\frac{2\cos t}{-3\sin 3t}\right|_{t=\frac{\pi}{6}}$$

$$= \frac{2\cos\left(\frac{\pi}{6}\right)}{-3\sin\left(3\cdot\frac{\pi}{6}\right)}$$

$$= \frac{2\cdot\sqrt{3}/2}{-3\cdot 1}$$

$$= -\frac{\sqrt{3}}{3}$$

And by symmetry the other crossings have either that slope or its negative, so

$$\pm\frac{\sqrt{3}}{3}$$

9. An ellipse can be expressed by the parametric equations

$$x(t) = a\cos t \qquad y(t) = b\sin t.$$

Use this parametrization to find the area of the ellipse.

In parametric curves, $\text{Area} = \int_{\alpha}^{\beta} y(t)\cdot x'(t)\,dt$

So here $\text{Area} = \int_0^{2\pi} (b\sin t)(-a\sin t)\,dt$

$$= -ab\int_0^{2\pi} \sin^2 t\,dt$$

$$= -ab\left[\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right]_0^{2\pi}$$

$$= -ab\left[\left(\tfrac{1}{2}(2\pi) - \tfrac{1}{4}\sin 2(2\pi)\right) - \left(\tfrac{1}{2}(0) - \tfrac{1}{4}\sin 2(0)\right)\right]$$

$$= -ab\,(\pi - 0 - 0 + 0)$$

$$= -\pi ab$$

Finally we discard the negative sign, which only reflects the fact that this parametrization traverses the ellipse backwards (right-to-left in quadrants I and II) or upside down (in quadrants III and IV).

$\therefore$ Area $= \pi ab$

10. Set up an integral for the length of the cardioid $r = 1 + \sin\theta$.

$$\frac{dr}{d\theta} = \cos\theta$$

$$\text{Length} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{(1+\sin\theta)^2 + (\cos\theta)^2}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{1 + 2\sin\theta + \underbrace{\sin^2\theta + \cos^2\theta}_{1}}\, d\theta$$

$$= \int_{0}^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$