Exam 3b Calc 2 4/1/2011

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. a) Write the first three terms in the sequence $\left\{\frac{n}{2n+1}\right\}$.

$\frac{n}{2n+1}$

$a_1 = \frac{(1)}{2(1)+1} = \frac{1}{3}$

$a_2 = \frac{(2)}{2(2)+1} = \frac{2}{5}$

$a_3 = \frac{(3)}{2(3)+1} = \frac{3}{7}$

Great!

b) Write the first three partial sums of the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$.

$\frac{n}{2n+1}$

$S_1 = \frac{(1)}{2(1)+1} = \frac{1}{3}$

$S_2 = \frac{(1)}{2(1)+1} + \frac{(2)}{2(2)+1} = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}$

$S_3 = \frac{(1)}{2(1)+1} + \frac{(2)}{2(2)+1} + \frac{(3)}{2(3)+1} = \frac{1}{3} + \frac{2}{5} + \frac{3}{7} = \frac{122}{105}$

2. Find the sum of the series $\frac{3}{4}-\frac{1}{2}+\frac{1}{3}-\frac{2}{9}+\frac{4}{27}-\ldots$

it is a geometric series

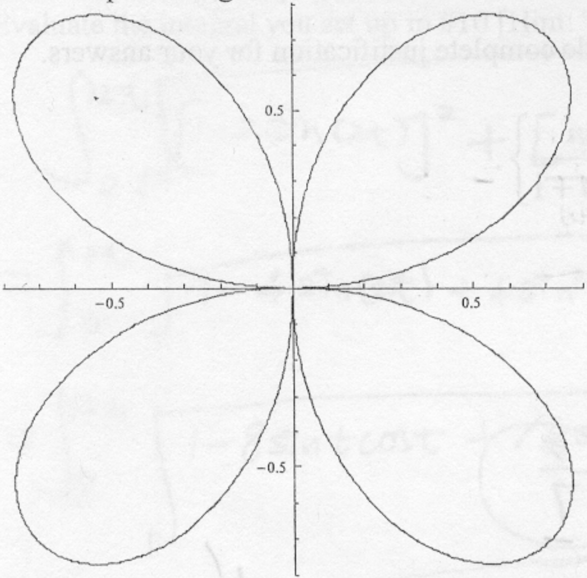
Excellent!

$$\text{Sum} = \frac{a}{1-r} \qquad a = \frac{3}{4}$$

$$r = -\frac{2}{3} \qquad |r| < 1 \text{ so it will work}$$

$$\frac{\frac{3}{4}}{1+2/3} = \frac{3/4}{5/3} \quad \frac{3}{4}\cdot\frac{3}{5} = \boxed{\frac{9}{20}}$$

3. Set up an integral for the area inside one loop of the curve $r = \sin 2\theta$.



$r = \sin 2\theta$

$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$

$= \frac{1}{2}\int_{\alpha}^{\beta} \sin^2(2\theta)\, d\theta$

4 loop $\Leftrightarrow \theta \in [0, 2\pi]$

1 loop $\Leftrightarrow \theta \in [0, \frac{1}{2}\pi]$

$\therefore \text{Area} = \frac{1}{2}\int_{0}^{\frac{\pi}{2}} \sin^2(2\theta)\, d\theta$

Great

4. Find a general solution to the differential equation $\dfrac{dy}{dx} = \dfrac{-x}{y}$.

$$\frac{dy}{dx} = \frac{-x}{y} \qquad dy = \frac{-x\,dx}{y} \Rightarrow y\,dy = -x\,dx$$

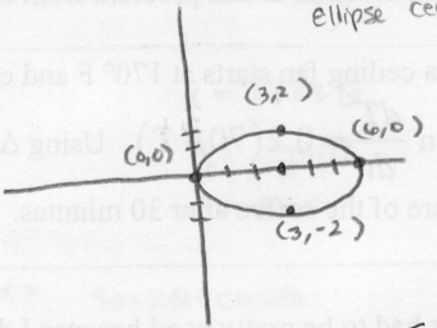
$$\int y\,dy = \int -x\,dx \Rightarrow \left(\frac{y^2}{2} = \frac{-x^2}{2} + C\right)\times 2$$

Nice!

$$y^2 = -x^2 + C \Rightarrow \underline{y^2 + x^2 = C} \text{ or } \underline{y^2 + x^2 = r^2}$$

5. Find an equation for the ellipse with vertices at (0,0), (6,0), (3,2), and (3,–2).

ellipse center (3,0)



$$\frac{(x-3)^2}{9} + \frac{y^2}{4} = 1$$

Excellent!

test

(0,0) $\frac{(0-3)^2}{9} + \frac{(0)^2}{4} = 1 + 0 = 1$ ✓

(6,0) $\frac{(6-3)^2}{9} + \frac{(0)^2}{4} = 1 + 0 = 1$ ✓

(3,2) $\frac{(3-3)^2}{9} + \frac{(2)^2}{4} = 0 + 1 = 1$ ✓

(3,-2) $\frac{(3-3)^2}{9} + \frac{(-2)^2}{4} = 0 + 1 = 1$ ✓

6. A cup of coffee starts at 160° F and cools according to the differential equation $\frac{dT}{dt} = 0.05(70 - T)$. Using $\Delta t = 5$ minutes, approximate the temperature of the coffee after 10 minutes.

Plug in T to find $\frac{dT}{dt}$

t	T	$\frac{dT}{dt}$	ΔT
0	160	-4.5	-22.5
5	137.5	-3.375	-16.875
10	$\boxed{120.625}$		

160 - 22.5 = 137.5

137.5 - 16.875 = 120.625

Great

The temperature after 10 minutes is approximately T= 120.625 °F

7. Biff is a Calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. This Euler's stuff is killing me. Look at this problem from our exam!"

> A paper cup of coffee sitting under a ceiling fan starts at 170° F and cools according to the differential equation $\dfrac{dT}{dt} = 0.2(70 - T)$. Using $\Delta t = 10$ minutes, approximate the temperature of the coffee after 30 minutes. Comment on the accuracy of your answer.

"So I worked it out, and said the answer had to be pretty good because I didn't have to round it off or anything. But when they gave the text back I got zero points for that part even though I did the number part right. Isn't math supposed to be about getting the right answer?"

Explain clearly to Biff what might have been a good comment on the accuracy of his answer.

The problem with Euler's method is that the larger the Δt value you use, the larger the stew in your answer. This is due to the fact that you are using the first tangent from $t=0$ all the way until $t=10$, thus the tangent will either be much steeper than the actual slope or much more horizontal than the actual slope

In your case there is a much steeper slope and the temperature become negative and then positive which is not going to happen in real life

Exactly!

t	T	$\frac{dT}{dt}$	ΔT
0	170	-20	-200
10	-30	20	200
20	170	-20	-200
30	-30	20	200

$\Big\}$ NOT GOING TO HAPPEN

Slope tan line

8. Find the ~~width~~ of the polar curve $r = \sin 2\theta$ (picture provided on #3).

$$\text{Slope} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

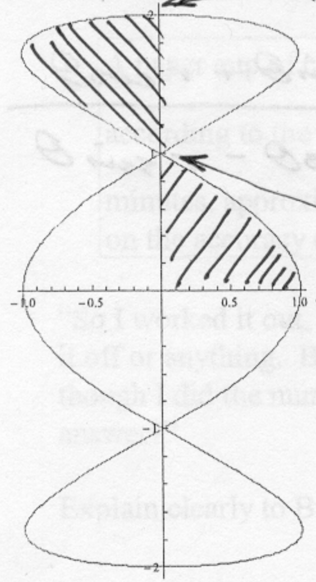
$$\frac{dr}{d\theta} = 2\cos(2\theta)$$

Good.

$$\frac{2\cos(2\theta)\cdot\sin\theta + \sin(2\theta)\cdot\cos\theta}{2\cos(2\theta)\cdot\cos\theta - \sin(2\theta)\sin\theta}$$

9. Find the area of the region enclosed by the curve with parametric equations

$$x(t) = \cos 3t \qquad y(t) = 2\sin t.$$



I can find the t for this point by letting $y = 2$, so

$$2 = 2\sin t$$
$$1 = \sin t$$
$$t = \pi/2$$

I can find this point's t by letting $y = 1$, so

$$1 = 2\sin t$$
$$1/2 = \sin t$$
$$t = \pi/6$$

This point has $x = 1$, so

$$1 = \cos 3t$$
$$0 = 3t$$
$$t = 0$$

So the part is traversed right-to-left, so it's area should be

$$-\int_0^{\pi/6} (2\sin t)(-3\sin 3t)\,dt$$

Then the part has its bottom traversed right-to-left, but top left-to-right, so the integral over that interval will give what's under the top minus what's under the bottom, whis is exactly the part we want, so

$$\int_{\pi/6}^{\pi/2} (2\sin t)(-3\sin 3t)\,dt$$

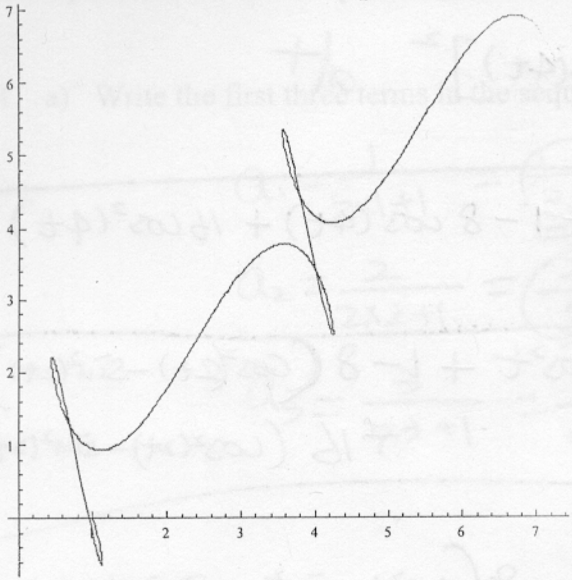
Overall there are 4 of each such part, so

$$\text{Total Area} = 4 \cdot \left[\int_{\pi/6}^{\pi/2} (2\sin t)(-3\sin 3t)\,dt - \int_0^{\pi/6} (2\sin t)(-3\sin 3t)\,dt\right]$$

10. Set up an integral for the length of the portion of the curve with parametric equations

$$x(t) = t + \cos 2t \qquad y(t) = t - \sin 4t$$

which extends from the point $(1,0)$ to the point $(2\pi + 1, 2\pi)$.



$x'(t) = 1 - 2\sin(2t)$

$y'(t) = 1 - 4\cos(4t)$

from $(1,0)$ to $(2\pi+1, 2\pi)$

On $(1,0)$

$\left.\begin{array}{l} 1 = t + \cos(2t) \\ 0 = t - \sin(4t) \end{array}\right\} t = 0$

On $(2\pi+1, 2\pi)$

$\left.\begin{array}{l} 2\pi+1 = t + \cos(2t) \\ 2\pi = t - \sin 4t \end{array}\right\} t = 2\pi$

Great

$\therefore t \in [0, 2\pi]$

$\therefore$ Arc Length $= \displaystyle\int_0^{2\pi} \sqrt{[1 - 2\sin(2t)]^2 + [1 - 4\cos(4t)]^2}\, dt$